TRACT III.

THE PRINCIPLES
OF
PENDENT BRIDGES,

WITH REFERENCE TO

THE PROPERTIES OF THE CATENARY APPLIED TO THE MENAI BRIDGE:

ALSO

A THEORETICAL INVESTIGATION OF

THE CATENARY.

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INTRODUCTION.

By the second report, "Holyhead's Roads and Harbour, 9th June 1810, ordered to be printed by the House of Commons," a cast-iron bridge was proposed by Mr. Rennie to be constructed over the Strait of Menai at Ynys y moch, estimated at 268,500*l.*; and by another report of May 30. 1811, a cast-iron bridge was proposed to be erected at the same place by Mr. Telford, estimated at 127,331*l.* These reports having fallen into the hands of the writer of this tract, he was led to an inquiry into the properties of bridges by suspension, and proposed by letters June 28. 1811, and subsequently, to those to whom the investigation of the subject of the Menai bridge had been principally entrusted, the erection of a chain-bridge over the Strait of Menai. He recommended such a mode of construction as adapted to remove the objections made to a bridge by the witnesses examined in respect to the navigation of the Strait, who anticipated much injury to vessels from baffling winds, in case such bridges as those which had been proposed should be built over the Strait. He also recommended it on account of its comparatively small cost, which he estimated at 34,600*l.*, also on account of the facility with which it might be constructed, and of the short time in which it might be rendered passable. But the following is a copy of the final paragraph of the report of February 18. 1819: "When it is remembered, that the first estimate prepared for Lord Colchester (when Chief-Secretary for Ireland) for building a cast-iron bridge across the Menai Strait, amounted to 268,500*l.*, your committee are of opinion, that the public stand greatly indebted to the industry of Mr. Telford, for having contrived a plan on so secure a principle for executing this work for the sum of 70,000*l.*"

Though the time which has been occupied in the author's design for a chain bridge over the Strait of Menai, and in the inquiry into the principles of its construction, may be rendered useless with respect to that bridge, perhaps it may afford, in the following pages, some information which may tend to prevent a very useful invention from again getting into disuse.

It will be seen by a reference to Scammozzi's Del Idea Archi. 1615, that bridges of this kind were in use in his time in Europe, although they are now held to be a modern invention, or derived from rope bridges used in the East Indies and South America.

Useful inventions are seldom lost to the public, if they require only ordinary skill to put them into practice; but the knowledge necessary to determine the properties of this kind of bridge had not been attained before the time of the Bernouilli*, and probably they were constructed upon false principles, and consequently were of short duration, and on that account the invention fell into disrepute.

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* See Johannis Bernouilli opera omnia, and Jacobi opera omnia, et Leib. et Bern. Com. Phil. et Math. a

For a span of 560 feet, and in a situation such as that of the Strait of Menai, where it is as important to maintain a good passage through, as to obtain one over it, a bridge of this kind is peculiarly suitable. But the strength of a hanging bridge is active, that of other bridges passive; and those who are of opinion that a bridge of this kind is to be constructed to have both these strengths, and also the advantages of insistent bridges in thoroughfares, may probably find themselves, when it is too late, in error.

The bridge proposed to be erected over the Strait of Menai, has been designed upon principles different from those demonstrated in this tract, and different conclusions have been come to. Upon the same data, by which 16 cables have been thought sufficient to sustain the roadway and the incumbrances of the proposed bridge, the author derives that 50 will be necessary. But as a gentleman of eminence has been selected to erect it, and as his design has been approved by a select committee of the

House of Commons*, the author, judging from authority, is fearful that what he has assumed, or what he has deduced, is not true.

The exposition of the properties of the catenary, from which alone the true principles of constructing bridges by suspension can be derived, and the application of those properties to such a practical case as that of the Menai bridge may afford to architects, who may be employed in the erection of such works, the means of determining, in as simple a mode as possible, the relative proportions of the parts of them.† The author has attempted to do this, and hopes there will be found as few defects as might have been expected from the nature of the inquiry.

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* The evidence of Mr. Rennie before the Committee is of a general nature. He says, "I have no doubt of the practicability of constructing such a bridge; and I presume that Mr. Telford has taken care to have all the parts sufficiently strong, and so well connected together, as to be able to sustain the weight he has calculated it to bear; but I have not made any calculations of the actual strength of the bridge which Mr. Telford proposes."

† In the last number (20) of "The Journal of Science, &c." edited at the Royal Institution, is a letter from Mr. Davies Gilbert, "On some properties of the Catenarian Curve, with references to bridges by suspension.

NATURAL BRIDGE

BY

SUSPENSION.

THE Chain and Rope Bridge is an invention of mountainous countries, and of very early times. The spider, travelling from tree to tree on a slender bridge of this kind, of her own manufacture, may have suggested it; and, perhaps, the subject may be best illustrated by a reference to this bridge-builder of nature's own tuition. The spider, about to construct a bridge of this kind, turns one of her nipples, which she can raise and depress at will, to the wind, and darts a thread with an elevation and force, and of a thickness proportionate to the distance of the place to which the projected end is intended to be attached: the one end of the thread she then fixes, and the other end becomes fixed accidentally, according to the direction of the wind. The thread in gravitating and stretching into the funicular curve, assumes a thickness, as will be shewn hereafter, in proportion to the secant of the angle made at the point of the curve between a horizontal line and the tangent at that point; that is, thinnest at the apex, gradually increasing in thickness to the points of suspension, accommodating its strength economically to the tensions at the several points of the curve. The bridge being now constructed, the spider begins to descend; and, by observing her progress, it will be seen that the angle (made by the tangent at the point of the curve, with a horizontal line, where the insect is) becomes increased by her weight, and the tension, in respect to the catenarian form, there consequently reduced; and also throughout the curve on that side the apex; and, therefore, the ordinate of the curve is always shorter than the corresponding ordinate on the other side, where there is no incumbent weight, and where the tension, in respect of the catenarian form, is increased.

Thus the bridge adjusts itself, first by proportioning its thickness to a state of rest at a curve, where the tension at the points of suspension is a minimum; and, secondly, by varying its form when moved, to resist the force of the mover with a like economy of strength.

The following table comprehends the most comprehensive account of bridges by suspension, which, from a reference to many volumes, could be obtained.

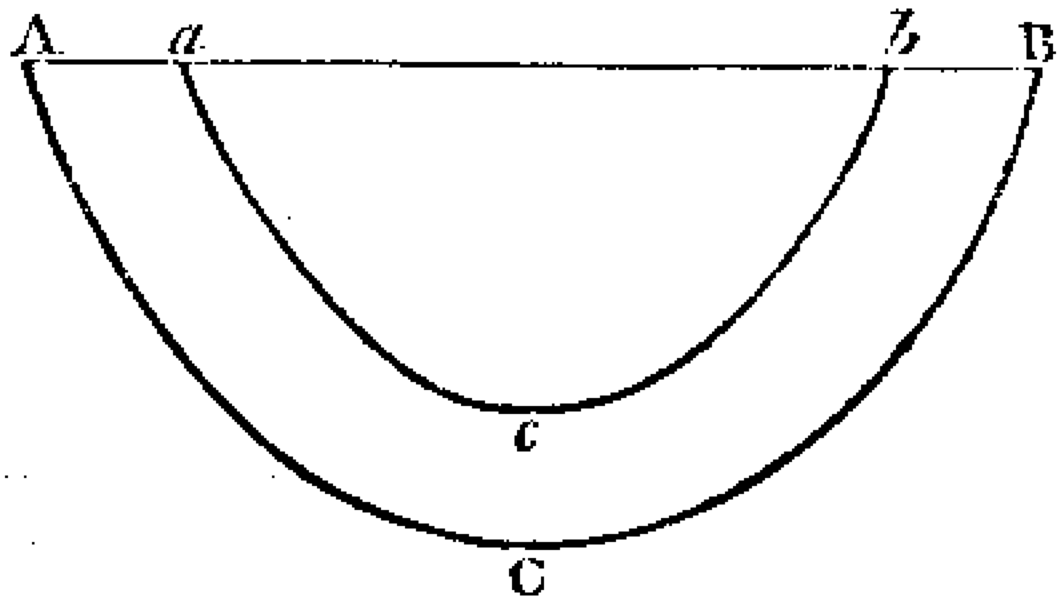
PENDENT BRIDGES.

Name of the Bridge.	River.	Where.	Length of the chord.	Versed Sine.	Thickness.	Material.	
		Junnan China	20 Chinese perches or 200 cubits		—	iron chains	65 A. D. Emperor Mingus. See Kircher's China illustrata, and Ogilby's China, 1669.
Chuka-cha-zum	Tchintchieu	18 miles from Murichon	140 feet		—	chains	See Turner's Tibet.
Apurima	Cleft in a Mountain	Andaquailas	720 feet		—	ropes	See Frezier's Voyage to the South Sea, 1712, 1713, and 1714.
	Alucrinda, or Ganges	Serinagur	240 feet		—	ropes	See Daniel's Views in Hindostan, No. 23, fourth series.
Selo-cha-zum	Tchintchieu	Chuka Mountain	70 feet		—	chains	See Turner's Tibet.
	J'hoola	Rampore	300 feet		—	rope	See Frazer's Tour to the snowy range of the Himala, and to the sources of the Ganges and Jumna.
	Arratis	Stoffingen in Switzerland	112 varchi*		—	wood linked forming a chain	See Scamozzi Della Archit. lib. 8. cap. 23. (There are many between Trento and Inspruck, as at Lavis, and over the Lisarco, at Cordau, at Leman, and Persenum, and eight or nine others.
Garbichen, near	Lucerna in	Switzerland	200 varchi*		—	woodlinked	* Cannot ascertain the length of a varco, suppose it to mean a pace.
	Eno	Copstan	150 varchi*		—	chains	There is also a chain bridge over the Pegnitz, at Nuremberg, in Franconia.

Beraun	Misa	Bohemia	116 feet Eng.		—	double chains	
Chain Bridge	Merrimack	Newbury Port	244		—	chains	See Monthly Magazine, May, 1811, page 361. Built by Mr. Templeman, and cost 25,000 dollars. Also see Pope's Bridges.
Cumberland	Potomac	Maryland	130		1¼ inch bar	chains	See Pope on Bridge Architecture, published at New York, 1811.
	Potowmac	Federal City	130		1¼ bar	chains	
	Brandywine	Wilmington	145		1⅜ bar	chains	
Brownsville	Monongahela	Fayette County	120		1¼ bar	chains	
Ditto	Ditto	Ditto	112		1¼ bar	chains	
	Schuykill	Philadelphia	140		1½ squ. bar	chains	
	Tees	Middleton	70			chains	Hutchinson's Antiquities of Durham.
	Tweed	Dryburgh	261			chains	See Times Newspaper, Aug. 13. 1817.
Union	Tweed	Berwick	432 between abutments 590 suspending chains from the points of junction		—	chains	Captain Brown, 1819. Cost 5000*l*. Weight of the platform, chains, &c. 160 tons. See Times Newspaper, August, 1820.
Menai	Menai Strait	Bangor	560	37	12 square inches in section	½ inch iron rods bound together forming a cable	Telford, 1820, now in progress; suspended weight 489 tons; connected rods weigh 117lbs. per yard; 16 cables containing in section 192 square inches. (See third Report Menai Bridge, ordered to be printed by the House of Commons, 8th February and 29th April, 1819.

OBSERVATIONS ON THE STRENGTH AND STRESS OF CHAINS FORMING CATENARIES.

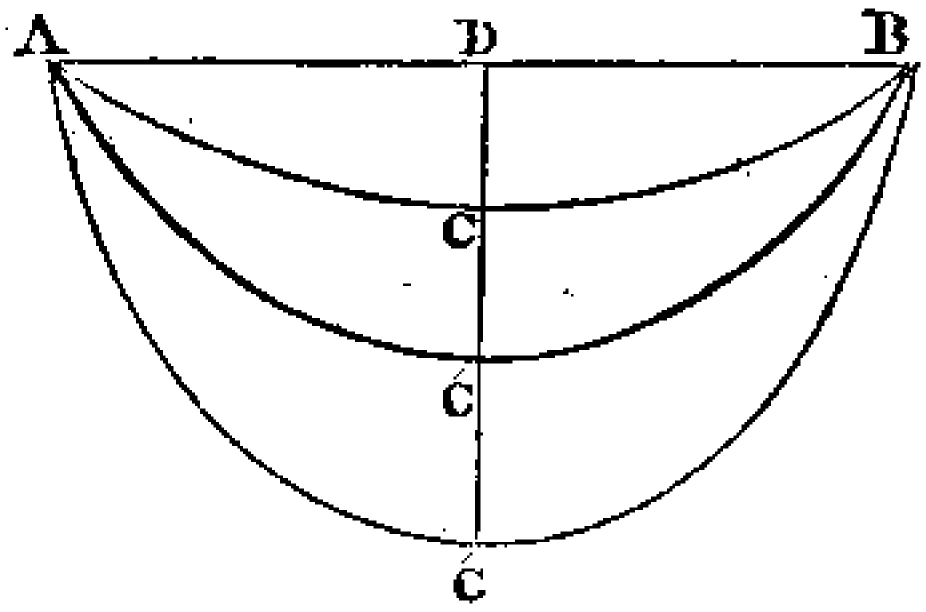
Let ACB and *acb* be similar catenaries, formed by chains uniform, and equally heavy in every point.



If the chain *acb* be able to support a given weight, another chain ACB of the same material, uniform, and equally heavy in every point with the former, may be taken so *long* as to be able to bear just its own weight, and any *longer* chain cannot support itself, but must break by its own weight. For the stress being as the length, and the strength remaining constant, the position is manifest.

Again, let the catenaries A C B,

$AC'B$, $AC''B$, be dissimilar with the same distance AB between their points of suspension, formed by chains uniform and equally heavy in every point.



If the chain $AC'B$ support its greatest weight, that is, assumes a figure of maximum strength, no other chains ACB, $AC''B$, of the same material, uniform and equally heavy in every point with the former, can be taken either longer or shorter, let down or drawn up, so as to be able to bear that weight, but must break.

If it be taken longer or shorter, and still required to bear the same weight, it must be increased in the area of its section, for the stress or tension increasing as the angle ABC diminishes, and increasing with the length of the chain beyond the maximum limit; it follows, therefore, that the capability of bearing weight increases as the ratio between the tension and ordinate diminishes, that is, when the tension is least, for the ordinate is supposed constant, the position is obvious.

Therefore, at some angle the stress will be the least quantity, and it is manifest that it cannot be when the angle = 0°, for then it cannot sustain its own weight; for a chain cannot by any force be drawn into a horizontal line, nor can it be when the angle is 90°, for then the length and stress are infinite, and therefore the least quantity must lie between 0° and 90°.

In any case, if a chain will just support its own weight, no other similar chain in form, and of the same specific gravity, however increased in the area of its section, will bear more than its own weight in the same figure.

In any catenary, when the chain forming it supports a given weight, and that weight is uniformly distributed and suspended from the chain, and is of the same matter as the chain, that is, suppose pieces of the same chain be suspended from each link, then their ends must form some line, and that line must be concave to the horizon, convex to the horizon, or a horizontal line. The line can only be a horizontal line when the tension at every point of the curve, less the absciss of the curve at that point equals a constant quantity, and as such pieces represent the tensions, the line can only be horizontal when the weight at the apex is a piece in length equal to the constant quantity, or equal in weight to a piece of that length. If, then, the piece at the apex be less than the constant quantity, or the weight be less than the weight of that piece, the line will be concave to the horizon; on the contrary, convex.

Mr. Emerson says, " When the thick-" ness is very small, speaking in the case " of an arch, (page 330., Doct. Flux.) whose " intrados is a catenary, it is of the same " thickness every where; consequently a " *heavy flexible* line put into this figure " would support itself." This appears an impossible case; for the tension is in a constant ratio to the secant of the angle;

therefore, the angle must vanish to produce an equal thickness, hence a mathematical should be substituted for a heavy flexible line, that is, one which has no weight.

Nature, in respect to the tenacity of the material she has given for chain bridges, and in respect to the limits as to form which she has prescribed for them, has measured her bounty by the possible wants of man; and she has left it to the industry and talents of any one to display that bounty and define her laws, in respect to the one and the other, which the author has attempted to do. But she has left it only to a few to display her bounty in practice, and, inasmuch as they approach those limits which she has prescribed, they bring into view her glory, and extend their own fame. It is for them duly to study the principles which are herein attempted to be sought after, and which hold in all machines and in all animal bodies, and to keep within the certain limits, " in regard to magnitude, which " not only in all machines and artificial " structures, but also in natural ones, sup" posing them made of the same matter, "and in the same proportion of parts, "neither art nor even nature herself can "go beyond."

If the weight of the roadway and incumbrances be omitted in the calculation, and the angles, at the points of suspension, be taken at 7° 10′ as the maximum angle or hill for a road-way, and 224000 ounces the practical strength per inch square of iron, being one-fourth of the breaking strength, and 54.083 ounces the weight of a foot linear of iron inch-square, then the ordinate becomes 515.36 * feet; and if the ordinate be given, viz. 250, then the smallest angle becomes 2° 44′, affording an ascent so small as to be unobjectionable in any roadway; so that nature has limited, under these circumstances, the span of a bridge by suspension to 1030.72 feet †, at a maximum

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* See formulæ $y = \frac{f\,m}{g}$ cos. ϕ log. tang. $(45^\circ + \frac{1}{2}\,\phi)$ and cos. ϕ log. tang. $(45^\circ + \frac{1}{2}\,\varphi = \frac{g\,y}{f\,m}$

† The distance between the points of suspension of the proposed bridge at Runcorn is 1000 feet, the angle of the catenary, with a horizontal line, 11°,,15′, consequently the deflection would be 100 feet, which, added

angle for a roadway, and at a span of 500 feet has fixed the minimum angle at 2° 44′ when iron is the material; supposing the allowance of three-fourths of the breaking strength to compensate for the weight of the roadway and its contingent incumbrances; which, though the author is not so daring as to think a sufficient allowance, yet there are others, who are of a less timid nature and of greater eminence than he can pretend to, who do.

In a chain-bridge, the greater the deflection, or as the angle made with a perpendicular line diminishes to a certain limit, the greater the proportion of its strength to its stress; and, like small animals, it will carry so much more than its own weight as the deflection increases to that limit, and will run faster, jump farther, and perform any motion quicker when acted upon; that is, its vibration will increase according to

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to the 70 feet above high-water mark, would make the props 170 feet above that level, besides their depth in the water. (See Barlow, Strength and Stress of Timber, Append.)

the deflection, to a certain limit, and beyond that limit its motion will be so rapid that it will break to pieces. *

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* Mr. Emerson has these scholia. "It is impossible that mechanic engines can be increased to any bigness. For when they arrive at a particular size, their several parts will break, and fall asunder by their weight.

"It is also impossible for Nature to produce animals above a particular size at pleasure, except some sort of matter could be found to make the bones of, which is so much harder and stronger than any hitherto known; or else that the proportion of the parts be so much altered, and the bones and muscles made thicker in proportion, which will make the animal distorted, and of a monstrous figure, and not capable of performing any proper actions. And being made similar, and of common matter, they will not be able to stand or move, but, being burdened by their own weight, must fall down.

"On the contrary, when the size of animals is diminished, their strength is not diminished in the same proportion as their weight, and therefore a small animal will carry far more than its own weight, whilst a great one cannot carry so much as its own weight. And hence it is that small animals are more active, will run faster, jump farther, or perform any motion quicker for their weight, than larger animals; for the lesser the animal, the greater the proportion of the strength to the stress. And Nature seems to have no bounds as to the smallness of animals, at least, in regard to their weight.

"Neither can any two unequal and similar machines resist any violence alike, or in the same proportion, but the greater will be more hurt than the lesser. And the

The same will hold as to the parts of a chain-bridge as to the parts of an animal. The motion in a chain-bridge is more rapid, as the angle with a horizontal line diminishes, or as the part is more distant from the point of suspension; so the motion of animals is quickest in the extremities, where the parts are the smallest, and most distant from the point of the suspension of the limb. Neither can two similar chain-bridges, composed of chains similar, but unequal as to area in section, resist any violence alike, or in the same proportion; for their flexibility or power of adjustment by their motion will be diminished; and the thicker will be more hurt than the thinner. For example, a chain bridge, one foot wide, containing four chains, inch square, will be stronger than a similar bridge containing one chain, two inches square.

Since it is desirable in a bridge by sus-

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same is true of animals; for large animals, by falling, break their bones, whilst lesser ones, falling lower, receive no damage. Thus a cat may fall two or three yards high and be no worse, and an ant from the top of a tower."

pension, that the passage should be as free as possible, that the vibration should be the least possible as to the whole; and that the flexibility of its parts should be the greatest possible, it follows that the roadway should be on the bearers; for, if the roadway be suspended and be divided into passages, as in the case of the proposed Menai Bridge, the way will be obstructed across by the suspending rods separating the coming, going, and footways; and it also follows that the angle made with a horizontal line should be the least possible; for the tighter the chain, the less its vibration as to the whole, and the chains should be as thin as possible; for the thinner the chains the greater the flexibility and power of self-adjustment as to the parts; and the thinner the more numerous; and, to use a vulgar adage, the fewer eggs will be in one basket. If flexible rods be substituted for chains, then the catenary ceases to be the curve from which the properties of such bridges are to be derived. The enquiry then falls under the case of a beam deflected by a weight, to determine how much more it will bear; which is referable to the elastic curve, and not to a catenary, and is not a question of direct tension, but one of tension and compression; and if such rods be used, they ought to be made in a curve, and not bent into one by force, or their own weight; they would then have the full benefit of their rigidity, as well as of their tension.*

Again, as it is desirable that the chord line should be as short as possible, is is improper to have a pyramid or prop with the apex over the middle of the base; for, if the ties outward are sufficient to resist the draft inward, that is, to keep the props perpendicular, they will be kept perpendicular whether the section of the props be an isosceles triangle or a right-angled triangle; and it is to be kept in mind that the tension and weight increase, (the angles at the points of suspension being constant,) with the length of the chord of the arch; and even if the angle be varied to reduce the tension in the limit applicable to practice, the weight and expense are necessarily increased, and therefore it is contrary to good judgment to increase the chord line.

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* Mr. Barlow (Strength and Stress of Timber, p. 244.) computed the strength of the proposed bridge at Runcorn with reference to the catenary: the author has on his authority so computed the strength of the bridge over the Strait of Menai.

Again, as the height of the points of suspension are increased, so is the power of the lateral draft of the chain to pull the piers over; and, therefore, for this cause also the pier should be the section of a right-angled triangle; that is, the side of the pier should be vertical next the bridge, for then the angle may be the same, and the pier lower.

Again, in the case where the weight suspended is such as to produce a horizontal line below the apex, it must be a compact mass of matter homogeneous with that of the chain in depth at the apex equal to the constant quantity, which, in a chain-bridge of 580 feet chord, and at angles at the points of suspension 14° 53′, would be 1065.57 feet. As the constant quantity is always equal to the radius of curvature at the apex, it is not possible a case can arise in which a horizontal roadway can be suspended, and the catenary retain its figure.

ONE FOOT LONG, INCH SQUARE.

Material.	Weight in Ounces.	Weight in Ounces which would tear asunder an Inch square of the Material in the First Column, in direct Tension.	From	Transverse supported at each End and Weight in the Middle, in Ounces.	From
		Tons. *Ounces.*			
Wrought Iron	$\frac{7788}{144}$ = 54.083	25 × 35840 = 896000	Experiments by Capt. T. Brown, Mill-Wall, Poplar.		
Cast Iron	$\frac{7207}{144}$ = 50.0486	7.264 × 35840 = 260341.76		*lbs.* 2910 × 16 = 46560	Experiments at Colebrook Dale, 1795.
Hempen Rope	$\frac{1236}{144}$ = 8.587	*lbs.* 5264 × 16 = 84224	Reports, Huddart and Co., common Staple taken at 8 inches in circumference.		
Ash	$\frac{600}{144}$ = 4.166	17000 × 16 = 272000	Experiments by Mr. Barlow, Royal Military Academy.		
Fir	$\frac{600}{144}$ = 4.166	12000 × 16 = 192000		460 × 16 = 7360	Mr. Bankes, Power of Machines.
Beech	$\frac{700}{144}$ = 4.861	11500 × 16 = 184000			
Oak	$\frac{845}{144}$ = 5.868	10000 × 16 = 160000		660 × 16 = 10560	

[*] Material.	Weight of an English Cubic Foot in Ounces.	Weight in Ounces which would crush a Cube of 1.968 Inches English.
Stone from the Quarry of Fourneaux, near Saumur, used for the columns of the Church of Toussaints d'Angers, diameter 30 centimets height, 7.8 métres - - -	2571	386000
Stone from the Quarry of Bagneux, used for the pillars of the dome of the Church of St. Génèviève - -	2203	216720
Stone from the Quarry of Mont Souris, used in the dome of St. Génèviève - -	2045	108570
Stone from the Quarry of Saillancourt, used in the Bridge of Neuilli - - -	2408	105640
		Weight which would crush an Inch Cube in the Direction of its Fibres,
Elm - - -	500	1200 lbs.
Fir - - -	600	2000 lbs.
Oak - - -	845	4000 lbs.
		Weight which would crush $\frac{1}{4}$ of an Inch Cube,
Cast Iron - -	7207	448000 lbs. Avoir.

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[*] See Gauthey, Traité des Ponts.

Material.	Weight of a Cube Foot.
	Ounces about
Stone, such as Portland,	2500
Limestone, - -	3200
Marble and Granites, -	2700
Brick, - - -	2000
Chalk, - - -	2500
Clay, - - -	2200
Sand, - - -	1500
Water, - - -	1000
Gravel - - -	1750
Mortar, - - -	1700
Coak, - - -	750
Coal, - - -	1300
Charcoal, - - -	450
Wrought Iron - -	7788
Cast Iron - - -	7207
Lead - - -	11400
Copper - - -	8785

TABLE containing the Velocity and Force of Wind, according to their common Appellations.

Velocity of the Wind.		Perpendicular Force on One Foot Area in Pounds Avoirdupois.	Common Appellations of the Force of Winds.
les in 1 Hour.	Feet in One Second.		
1	1.47	.005	Hardly perceptible.
2	2.93	.020	Just perceptible.
3	4.40	.044	
4	5.87	.079	Gentle pleasant wind.
5	7.33	.123	
10	14.67	.492	Pleasant brisk gale.
15	22.00	1.107	
20	29.34	1.968	Very brisk.
25	36.67	3.075	
30	44.01	4.429	High winds.
35	51.34	6,027	
40	58.68	7.873	Very high.
45	66.01	9.963	
50	73.35	12.300	A storm, or tempest.
60	88.02	17.715	A great storm.
80	117.36	31.490	A hurricane.
100	146.70	49.200	A hurricane that tears up trees, carries buildings before it, &c.

[*See Smeaton's Experimental Enquiry, Wind and Water.*

TO DETERMINE THE STRENGTH OF THE BRIDGE OVER THE STRAIT OF MENAI. — THE ROADWAY SUSPENDED FROM IRON CABLES, UPON THE SUPPOSITION THAT THE WHOLE FORMS A CATENARY.

Let $x = 37$ feet, the absciss.* Fig. 1.

$y = 280$ feet, the ordinate.*

Then $z = 283.233$, the length of the cable between a point of suspension and the apex, by calculation.

$c = 1065.57$ the constant force at the apex of the curve.

$\varphi = 14^\circ\ 53'$ the angle between the ordinate and the tangent at the point of suspension.

$t = 1102.64$ the tension of the cable at the same point.

$g = 649$ lbs. avoirdupois, the weight of the cable per foot; for there are to be 16 cables, the area of whose collected sections (12 square inches each) amount to 192 square inches: as 144 : 7788 oz. (spec. grav.) : : 192 : 649 lbs.

$w = 587.5$ tons, being half the weight to be suspended; arising as follows:

	Tons.
Weight of the cable, $2z \times 649$ lbs. $=$	164

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* See 3d Report, House of Commons, April 29. 1819.

	Tons.
Brought up	164
The force of hurricane winds, being 49.2 lbs. per foot superficial on the road; viz. $560 \times 30 = 16800$ surface of the road: 16800×49.2 lbs. $= 369$ tons.	369
Oxen, infantry, cavalry,[*] or other things passing at one time - -	300
Weight of the road to be suspended. See the Report. - - - - - -	342
Whole weight suspended from the two points - - - - - - - -	1175

$s = 738.5$ tons, the strength of the cables collected; derived as follows:

A bar of iron, inch square, acted upon in a perpendicular direction, at a mean of eight experiments, made at Captain Browne's Manufactory, Mill-Wall, Poplar, broke with 25 tons. One-fourth the breaking strength[†], viz. $6\frac{1}{4}$ tons, is held to be the measure to be used in calculations of strength for practical purposes.

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[*] Though by 49 sec. of the act 59 Geo. 3. cap. 48. the number of oxen and neat cattle is limited to 20 on the bridge at one time, yet the number of cavalry and infantry is not nor could be limited. It cannot be supposed that 5000 soldiers on a forced march would be delayed five hours in passing this bridge.

[†] Mr. Emerson thinks one-fourth the breaking strength should be taken as the practical strength. Mr. Rennie (3d Report, Menai Bridge, 29th April, 1819, page 27.) says, "I have before stated, that I consider a bridge,

But the 16 cables, the area of whose collected sections amounts to 192 * square inches, are to be composed of half an inch square each, to be welded in one length; they are then to be secured together at every 5 feet by bucklings, and wrapped in flannel well saturated with a composition of rosin and bees' wax, and further bound together with iron wire about one-tenth of an inch in diameter. †

By the experiments made to ascertain the relation between the strength of a cable of about the same number of fibres as there are half-inch square bars in these, to the sum of the strengths of its fibres, it has been found to be reduced as $1040 : 640$; then, as $1040 : 640 :: 6\frac{1}{4} : 3.846$; and $192 \times 3.846 = 738.5$. Now, supposing the suspended weight not to alter the catenarian form, we have at the points of suspension, $z : t :: w : \frac{t}{z} \times w = 3.8933 \times 587.5 = 2287.32$ tons.

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such as this is, should be above four times as strong as a mere theoretical calculation gives it.

* It appears by calculation, that t is always equal to the secant of the angle ϕ ($c =$ rad.) Therefore, if 192 square inches be the area of the section at the apex, the area of the section at 5° should be 192.73; at 10° = 194.96, and at the points of suspension where the angle is 14° 53′ 198.66 inches square and not of an uniform thickness as intended.

† See Barlow's Appendix, Essay on Strength and Stress of Timber.

But the strength of the cables is only 738.5 tons, leaving a deficiency in strength of 1548.82 tons nearly at each point of suspension: as 738.5 : 16 :: 2287.32 : 49.558.

We have at the apex,

as $z : c :: w : \frac{c}{z} \times w = \frac{1065.57}{283.233} \times 587.5 = 2210.27$

as 738.5 : 16 : : 2210.27 : 47.88

or as sec $\angle\,\varphi$: rad. : : 49.558 : 47.88.

Hence, there should be nearly 50 * cables instead of 16 at the points of suspension, and nearly 48 at the apex.

If what has been before advanced can be relied upon, a bridge so constructed cannot be called safe.

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* It is plain that this mode of calculation, nevertheless, brings out the number of cables too few by many because instead of the cables weighing 164 tons as taken in w, they will weigh 508 tons.

In order to make the comparison, the same mode will be followed in respect to a bridge where the roadway is on the cables.

EXAMPLE, THE ROADWAY UPON CHAINS.

If the ascent be taken 1 in 8 * at the points of suspension, then $\varphi = 7° 10'$ and $\frac{l}{z} = 8.0156$, and $2w$ be taken at 1175 − (342 − 46.9 †) = 879.9, then $w = 439.95$ and $8.0156w = 3526.5$. Now instead of cables, if chain bars, as proposed by the author in his design, be used, then the strength will be $6\frac{1}{4} \times 192 = 1200$ 1200 : 16 : : 3526.5 : 47, and there will be required only 47 bars of iron, each Fig. 2&3.

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* "On the Welsh portion of the great road from Holyhead through North Wales to Shrewsbury, the inclinations were formerly, in many instances, as much as 1 in 6, 7, 8, 9, 10." See page 54., Report, Highways, June 25th, 1819. If 1 in 6 mean 1 perpendicular to 6 ascending, then
1 in 6 = 9°„35′; 1 in 7 = 8°„13′; 1 in 8 = 7°„10′; 1 in 9 = 6°„23′; 1 in 10 = 5°„44′.

† 46.9 tons are deducted from the weight of the suspended road for the platform which will be wanted in both cases.

of equal area, to one of the proposed cables * at the points of suspension, and the props may be 20 feet lower; and though at the points of suspension the ascent is 1 perpendicular to 8 ascending, the average will be 1 to nearly 16. In the new road from Shrewsbury to Holyhead, improved under the direction of the parliamentary commissioners, where "to the superior science" exhibited, the Select Committee of the House of Commons say in their Report "high praise is due," the ascent for a length of 600 feet is 1 to 17. (See page 55, Report, Highways of the Kingdom, 25th June, 1819, and page 9., 18th July, 1820.)

The excess of the cost (70,000*l*.) of the bridge recommended by the Committee of the House of Commons above that (34,600*l*.) recommended by the author, arises, 1st, because there are not in the latter any suspending rods nor bearers for the platform of the road; that resting upon the chain-bars. 2d, He proposed to continue chain-bridges to the shores from the props, instead of erecting arches from them; for chain-bars must be continued of the same tension to the shores to resist the horizontal drawing as ties, and therefore they may as well be used also for bearers. 3d, His bridge is not required to be so wide for the same thoroughfare, nor the props required to be so high; for if there be only 16 cables in the proposed bridge (instead of 50 which the author deduces to be required), each occupying a space of 4 inches, there must be a space of 5 feet 4 inches occupied where there cannot be any thoroughfare, and, of course, the smaller the angles at the points of suspension the lower the props.

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* To determine the height of the props above the apex of the curve, see the table at 7° 10′, then

as $\text{Tab. } y : \text{Tab. } x :: 280 : \frac{\text{Tab. } x}{\text{Tab. } y} \times 280 = 17.5726 =$

the absciss of the curve.

Having now investigated the manner of the construction of the Menai bridge, as recommended by the Committee of the House of Commons, as well by reference to a natural pendent bridge, as by calculation from principles derived from it; it remains now to forget the questions which were put to the witnesses who were examined relative to the strength of the bridge proposed to be erected over the Strait of Menai, and to state the questions which it may be supposed were intended to have been put, with the data, and to deduce the answer which would probably have been received, in order to give examples of the use of the investigation of the properties of the catenarian curve, and of the table in the subsequent part of this tract.

Formulæ * to determine the number of square inches in the collective sections of the chains of a penden bridge, when the weight of the road, the materials of which the chains are made, and any two of th five quantities ϕ, c, x, z, or y †, are given.

Put n for the number of square inches in the collective sections of the chains; g for the weight of a part of the material of which th chains are made, measuring one inch square, by one foot long; f for the weight or tension, which would just overcome the force cohesion in a like piece of the same material; w for the weight of the road upon each linear foot of the chains:

When	c and ϕ	z and ϕ	x and ϕ	y and ϕ	z and c	Are given Quantities.
n	$=\dfrac{w}{\dfrac{f}{c\,\text{sec.}\phi}-g}=$	$\dfrac{w}{\dfrac{f}{z\,\text{cosec.}\phi}-g}=$	$\dfrac{w}{\dfrac{f\,\text{ver.sin.}\phi}{x}-g}=$	$\dfrac{w}{\dfrac{fm}{y}\cos.\phi\,\text{log.tang.}(45^\circ+\frac{1}{2}\phi)-g}=$	$\dfrac{w}{\dfrac{f}{\sqrt{z^2+c^2}}-g}$	

When	x and c	x and z	c and y ‡	z and y §	x and y ‖	Are given Quantities.
n	$=\dfrac{w}{\dfrac{f}{c+x}-g}=$	$\dfrac{w}{\dfrac{2fx}{z^2+x^2}-g}=$	$\dfrac{w}{\dfrac{f}{c\,\text{sec.}\phi}-g}=$	$\dfrac{w}{\dfrac{f}{z\,\text{cosec.}\phi}-g}=$	$\dfrac{w}{\dfrac{f\,\text{ver.sin.}\phi}{x}-g}$	

Formulæ ¶ to determine the Catenary when n, g, f, w, and any one of the five quantities ϕ, c, x, z, and y † are given.

Given Quantities.	
ϕ	$c\,\text{sec.}\phi=\dfrac{f}{g+\dfrac{w}{n}}$ or $c=\dfrac{f\cos.\phi}{g+\dfrac{w}{n}}$
c	$\text{sec.}\phi=\dfrac{f}{c(g+\dfrac{w}{n})}$
x	$\dfrac{x}{\text{ver.sin.}\phi}=\dfrac{f}{g+\dfrac{w}{n}}$ or $\text{ver.sin.}\phi=\dfrac{x(g+\dfrac{w}{n})}{f}$
z	$z\,\text{cosec.}\phi=\dfrac{f}{g+\dfrac{w}{n}}$ or $\text{cosec.}\phi=\dfrac{f}{z(g+\dfrac{w}{n})}$
y	$\dfrac{y}{m\cos.\phi\,\text{log.tang.}(45^\circ+\frac{1}{2}\phi)}=\dfrac{f}{g+\dfrac{w}{n}}$ Or $\cos.\phi\,\text{log.tang.}(45+\frac{1}{2}\phi)=\dfrac{y(g+\dfrac{w}{n})}{mf}$ **

When the greatest catenary which can be formed of any given material is required, w and n will disappear, and then the formulæ will become

[Given] Quantities.	
ϕ	$c=\dfrac{f\cos.\phi}{g}$
c	$\text{sec.}\ \phi=\dfrac{f}{cg}$
x	$\text{ver.sin.}\phi=\dfrac{gx}{f}$
z	$\text{cosec.}\phi=\dfrac{f}{zg}$
y	$\cos.\phi\,\text{log.tang.}(45^\circ+\frac{1}{2}\phi)=\dfrac{yg}{mf}$

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* The value of n being $\dfrac{w}{\dfrac{f}{t}-g}$ these formulæ are obtained by substituting for t its values. See page 145, Tract III.

† These and m denote the same as in the Table of Formulæ, page 145.

‡ Find ϕ from the Formula $\text{log.tang}\,(45^\circ+\frac{1}{2}\phi)=\dfrac{y}{cm}$

§ Find ϕ from the formula $\cot.\phi\,\text{log.tang.}(45^\circ+\frac{1}{2}\phi)=\dfrac{y}{mz}$

‖ Find ϕ from the formula $\dfrac{\text{log.tang.}(45^\circ+\frac{1}{2}\ )}{\text{sec.}\phi\,\text{ver.sin.}\phi}=\dfrac{y}{mx}$

¶ The value of t being $\dfrac{f}{g+\dfrac{w}{n}}$ these formulæ are obtained by substituting this value for it. See page 145, Tract III.

** From which ϕ may be found by approximation.

EXAMPLE.

The navigation of the Strait of Menai Fig. 2 & 3. is to have a clear water-way of 500 feet at the level of highwater-mark, and a clear windway of 500 feet in width by 100 feet in height above that level.

The bridge is to be 30 feet wide.

The rise is not to exceed, in any part, one perpendicular to eight ascending, or an angle of 7°„10′.

The bridge is to be a pendent bridge, supported by stone piers.

The pendent bearers are to be wrought iron, hempen rope, oak, fir, or ash*, forming a catenarian curve, the weight of which materials and the strength of them in direct tension are to be taken according to the following table:—

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* How the wood is to be linked together, and with what metal, is not part of this enquiry; it is sufficient that bridges by suspension have been erected of pieces of wood linked together.

One foot long, inch square.	Weight in ounces.	One-fourth * the weight in ounces, which would draw asunder the material in the first column in direct tension.
Wrought Iron	$\frac{7788}{144} = 54.083$	$\frac{56000 \times 16}{4} = 224000$
Hempen Rope	$\frac{1236}{144} = 8.587$	$\frac{5264 \times 16}{4} = 21056$
Oak	$\frac{845}{144} = 5.868$	$\frac{10000 \times 16}{4} = 40000$
Fir	$\frac{600}{144} = 4.166$	$\frac{12000 \times 16}{4} = 48000$
Ash	$\frac{600}{144} = 4.166$	$\frac{17000 \times 16}{4} = 68000$

The weight in suspension to be taken as follows:—

	Ounces.
The road-way on the bearers to be 4 inches thick of some matter, weighing, per foot cube, 600 ounces - - -	200
Weight of the wind in hurricanes, per foot superficial, (49.2 × 16)	787

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* Mr. Emerson thinks one-fourth the breaking strength should be taken as the practical strength. See his Mechanics. Mr. Rennie, page 27, 3d Report, Menai Bridge, 29th April, 1819, says, "I have before stated that I consider a bridge such as this is, should be above four times as strong as a mere theoretical calculation gives it."

	Ounces.
Brought up	987
Weight traversing the road as cavalry, oxen, per foot superficial - - -	640[*]
Total weight, per foot superficial	1627

Required the area of the section of the pendent bearers?

Then let w = the whole weight which may be supported by the pendent bearers, per foot superficial.

g = the weight of one foot long, inch square, of the pendent bearers in ounces.

n = the number of square inches in the area of the collected sections of the pendent bearers.

f = the practical strength of iron or one-fourth of the weight in ounces, which would tear asunder an inch square of the pendent bearers.

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[*] That is, 40lbs. upon a foot superficial, or one man in four feet; in crowds, 8 men will stand in 9 feet.

Since g is the weight of a part of the pendent bearers measuring one inch square, and n the number of square inches in the sections of all the pendent bearers, we have

ng = the weight per foot superficial of the pendent bearers.
y = the ordinate of the catenary.
z = half the length of the catenary.
φ = the angle which the catenary makes with a horizontal line at each point of suspension.
t = the measure of the tension of the bearers at the points of suspension.
c = the measure of the same at the apex, which is constant.
$m = 2.302585$ = the number by which the common logarithm of any number must be multiplied to obtain the hyperbolic logarithm of the same number.

Then $w = 1627$, $g = 54.083$, or 8.587, or 5.868, or 4.166, according to the material used; $f = 224000$, or 21056, or 40000, or 48000, or 68000, according to the material used, and $y = 250$ * $\varphi = 7° 10'$.

Then by the formulæ in the subsequent part of this work, and by a table of sines, tangents, &c.

$$z = \frac{y \text{ Tang. } \phi}{m \text{ Log. Tang. } (45^{\circ} + \frac{1}{2}\phi)} = \frac{250 \times .1257}{m \text{ Log. Tang. } 48^{\circ},35'} = \frac{250 \times .1275.}{2.3025 \times .054464} = 250.655.$$

$$t = \frac{y}{m \text{ Cos. } \phi \text{ Log. Tang. } (45^{\circ} + \frac{1}{2}\phi)} = \frac{250}{2.3025 \times .99218 \times .054474} = 2009.25.$$

$$c = \frac{y}{m \text{ Log. Tang. } (45^{\circ} + \frac{1}{2}\phi)} = \frac{250}{2.3025 \times .054464} = 1993.56.$$

$$x = \frac{y \text{ Ver. Sin. } \phi}{m \text{ Cos. } \phi \text{ Log. Tang. } (45^{\circ} + \frac{1}{2}\phi)} = \frac{250 \times .0078126}{2.3025 \times .99218 \times .054464} = 15.69$$

$$n^{*} = \frac{w}{\frac{fm}{y} \text{cos. } \phi \text{ log. tan. } (45 + \frac{1}{2}\phi) - g} = \frac{1627}{111.488 - 54.083} = 28.342$$

These values are more easily obtained from the table of the dimensions of a catenary as follows :

As tab. y : tab. z : : the given y : the required $z = \frac{\text{tab. } z}{\text{tab. } y} \times y$

By the table $\frac{z}{y}$ at 7°„10′ = 1.00262 .·. z = 1.00262 × 250 = 250.655.

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[*] y is taken at 250 instead of 280; for it is better to reduce the span 60 feet, and have the side of the pier vertical, than to increase it 60 feet in order to have the pier batter 30 feet.

[†] See the formula when y and ϕ are given in this work.

As tab. y : tab. t :: the given y : the required $t = \frac{\text{tab. } t}{\text{tab. } y} \times y$

By the table $\frac{t}{y}$ at 7° 10′ $= 8.037 \therefore t = 8.037 \times 250 = 2009.25$.

As tab. y : tab. c : : the given y : the required $c = \frac{\text{tab. } c \times y}{\text{tab. } y}$ and because tab. $c = 1 \therefore c = \frac{y}{\text{tab. } y}$

By the table y at 7° 10′ $= .12541 \therefore c = \frac{250}{.12541} = 1993.56$.

As tab. y : tab. x :: the given y : the required $x = \frac{y}{\frac{\text{tab. } y}{\text{tab. } x}}$

By the table $\frac{y}{x}$ at 7° 10′ $= 15.927 \therefore x = \frac{250}{15.927} = 15.69$.

$$n^* = \frac{w}{\frac{f}{t} - g} = \frac{w}{\frac{224000}{2009.25} - g} = \frac{w}{111.489 - g} =$$

$$\frac{w}{111.489 - 54.083} = \frac{w}{57.406} = \frac{1627}{57.406} = 28.342 = n$$

the same as before.

The bridge being 30 feet wide, and one

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* See the formula when t is given.

chain being required to each foot superficial, there must be in the area of the collective sections of the chain-bars 30 × 28.342 = 850.26 square inches at each point of suspension.

Hence ($\frac{28.342}{12} = 2.362$) if the bearers be inch wide they must be 2.362 inches deep, making a sheet of iron 2.362 inches thick.

Let the bearers be rope; then

$$f = 21056 \text{ and } g = 8.587 \quad n = \frac{w}{\frac{f}{t} - g} = \frac{w}{\frac{21056}{2009.25} - g}$$

$$= \frac{w}{10.479 - 8.587} = \frac{w}{1.892} = \frac{1627}{1.892} = 859.93$$

Then $\frac{859.93}{12} = 71.66$

Hence, if the bearers be rope, they must be a compact sheet of rope, nearly 6 feet thick.

Let the bearers be oak, then

$$f = 40000. g = 5.868 \quad n = \frac{w}{\frac{f}{t} - g} = \frac{w}{\frac{40000}{2009.25} - g}$$

$$= \frac{w}{19.908 - 5.868} = \frac{w}{14.04} = \frac{1627}{14.04} = 115.88$$

Then $\frac{115.88}{12} = 9.65.$

Hence, if the bearers be oak, there must be in each superficial foot of the bridge 115.88 square inches of oak in section, and they may be 12 inches by $9\frac{3}{4}$ inches nearly each; and there must be 30 of them.

Let the bearers be fir, then

$$f = 48000. \quad g = 4.166 \quad n = \frac{w}{\frac{f}{t} - g} = \frac{w}{\frac{48000}{2009.25} - g}$$

$$= \frac{w}{24.089 - 4.166} = \frac{w}{19.923} = \frac{1627}{19.928} = 81.644$$

Then $\frac{81.644}{12} = 6.8$

Hence, if the bearers be fir, there must be in each foot superficial of the bridge 81.644 square inches of fir in section, and they may be 12 by $6\frac{2}{3}$ inches each; and there must be 30 of them.

Let the bearers be ash, then

$$f = 68000. \quad g = 4.166 \quad n = \frac{w}{\frac{f}{t} - g} = \frac{w}{\frac{68000}{2009.25} - g}$$

$$= \frac{w}{33.843 - 4.166} = \frac{w}{29.677} = \frac{1627}{29.677} = 54.82$$

Then $\frac{54.82}{12} = 4.5$

Hence, if the bearers be ash, there must be in each foot superficial of the bridge 54.82 square inches of ash in section, and they may be 12 by 4½ inches each; and there must be 30 of them.

It may be urged that it is not probable that a hurricane would take place when the bridge is covered with oxen or soldiery, and since one-fourth of the breaking-weight is taken as a measure of strength, that the hurricane-weight might be omitted in the calculation. On the other hand, it may be urged, something should be allowed for the operation of time in respect of wear and tear; and it must also be taken into the consideration that a bridge of this kind is of the nature of a cable, where the strength of the whole is not equal to the sum of the strengths of the fibres; for it could not be expected that the whole could be so nicely adjusted that every square inch of bar should perform its proportion of duty as to area, even if the metal were of one quality, and the manufacture uniform. If, therefore, in ordinary cases, one-fourth of the breaking-strength is held to be the measure of practical strength, still more, in a case so extraordinary, every allowance should be made for contingencies.

TO DETERMINE THE EXTRADOS OF REBUTMENT PIERS TO SUPPORT BRIDGES BY SUSPENSION, AND CONCERNING BEARING PIERS FOR SUCH BRIDGES.

Fig. 4. Let G C B be a suspended bridge, the points of suspension at the same level, and B I the height of the pier. I say if the angle of the extrados of the pier I B A be equal to L B I, the angle made by the catenary with B I, that B A will be the direction of the tying chains at B; and if the catenary be continued to F so that C E the absciss of the curve F B C at F = B I+D C and B I be continued till it intersect the ordinate E F of the catenary so continued at H, that H F will be the width of the base of the pier required, and B F the extrados, and B H F inverted will be represented by B I K, and B A will be a tangent to B K.

And the same will hold if the pier be built inwards, and push instead of pull, as in the case of a bridge over a deep valley, where the hills would be natural abutments.*

In like manner, if the bridge be a half-catenary, that is, a descending bridge from a high to a low bank, then the point C would be sustained by a pier whose extrados, C K, would be a catenary, similar to C G, the absciss C O = N C. Fig. 6.

For by the properties of the catenary, B will be equally fixed, whether it be the point of suspension, or whether F or any other point above it be the point; therefore B F supports the point B, and the catenary suspended from B. Now, if B F be inverted and represented by B K, and the figure B K be maintained by a rigid body, B K, and the point K be fixed, then there is no more reason why B should move in the one case than in the other; and I K = H F will be the base of the pier. Fig. 4.

But B is only a point; and it is manifest in a practical case, that it must have some width, and be some curve.

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* G C B is the seat of a camp-stool, and G N and B A the legs. Fig. 5.

Fig. 7. Let B V K be a parabola described by a body whose velocity is such at the angle D B C that it shall hit the plane K I in K, ascertained as above. It is manifest that some point V above B towards K will be the vertex of the parabola.

Fig. 4. By the same mode of reasoning, if the chain, F B, be let down from the fixed point, F, so as to assume the curve F *w* *v* B wherein B *v* *w* is similar to G C B, the catenary of the suspended bridge, then B *v* *w* F inverted would also be an extrados to the pier, differing very little from that of the parabola.*

Fig. 7. If the parabola B V K be substituted for the catenary B K (*fig.* 4.) there is no cause why the axis V O (*fig.* 7) should be drawn more towards B than to K, and therefore it must remain vertical, and B V K is the extrados of the pier required.†

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* The extrados of an abutment or rebutment pier is analogous to that of an arch. According to James Bernoulli, (Omn. Oper. page 449.) the catenary under certain conditions, which appear to the author applicable in this case, becomes a parabola. Dr. Gregory (Phil. Trans. 1697) only treats of the common catenary.

† Mr. Emerson has these scholia: "If a wall faces the

Let a body be projected from B at the height B I from the plane I K in the direction B T, so as to hit the plane in Fig. 8.

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wind, and if the section of it be a right-angled triangle, or the foreside be perpendicular to the horizon, and the backside terminated by a sloping plane intersecting the other plane in the top of the wall; such a wall will be equally strong in all its parts to resist the wind, if the parts of the wall cohere strongly together; but if it be built of loose materials, it is better to be convex on the backside in form of a parabola. If a wall is to support a bank of earth or any fluid body, it ought to be built concave, in form of a semi-cubical parabola, whose vertex is at the top of the wall; this is when the parts of the wall stick well together. But if the parts be loose, then a right line or sloping plane ought to be its figure. Such walls will be equally strong throughout.

"All spires of churches in the form of cones or pyramids are equally strong in all parts to resist the wind. But when the parts cohere not together, parabolic conoids are equally strong throughout.

"Likewise, if there be a pillar erected in form of the logarithmic curve, the assymptote being the axis, it cannot be crushed to pieces in one part sooner than in another, by its own weight.

"And if such a pillar be turned upside down, and suspended at the thick end in the air, it will be no sooner pulled asunder in one part than another by its own weight; and the case is the same if the small end be cut off, and instead of it a cylinder be added whose height is half the subtangent."— These observations are appli-

K: required P B, the height through which a falling body would acquire the projectile velocity, or the velocity at the point B. Then by the known properties of the parabola, draw K G perpendicular and B G horizontal, intersecting in G. Make B E = $\frac{BG}{4}$, and draw E N perpendicular, intersecting B T in N. Join B K, and draw B M perpendicular to B K, and with the centre, M, in the line B M describe a circle through B and N, intersecting I B continued in P. B P is the height through which a falling body will acquire the projectile velocity. Bisect P B in A, and with the centre, A, describe the circle through P and B, intersecting B T in H. Through H draw V L horizontal, intersecting P B in L. Make H V = H L, V is the vertex. From V draw V O perpendicular to V L and through H draw P F, intersecting V O in F;

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cable to the matter in question; but the author has not been able to come to the same conclusions in respect to some of them; which may arise from his not understanding their precise meaning.

or, which is the same, make the triangle H V F similar to B L H, B L, and H V homologous sides. V F is the focal distance, whence the parabola B V K.

To find H F.

Let the angle D B C = 7°.10′, the chord Fig. 4. G B = 500, the height of the point of suspension $= DC + CM = 15.697 + 100 = 115.697 = BI = BH = DE$.

$EC = DE + DC = 115.697 + 15.697 = 131.394.$

$$\text{Whence}\begin{cases} DC = 15.697 \\ DB = 250. \\ CE = 131.394. \end{cases}$$

To find the constant quantity at C.

As tab. y : tab. c : : D B : to the constant quantity $= c$; tab. c. being $= 1$, $c = \frac{DB}{\text{tab. } y.} = \frac{250}{.12541} = 1993.56.$

Then as c : tab. c : : C E : tab. $x = \frac{CE}{c} = \frac{131.394}{1993.56} = \frac{x}{c}$ in the table $= x$ in the table $=$ .065912, which corresponds with 20°.15′ in the table.*

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* $\underline{131.394} = 2.1185755$ log.
$1993.46 = \underline{3.2996076}$ log.
$20^\circ 15' = .065912 = \bar{2}.8189679$ log.

Fig. 4. As tab. x : tab. y : : C E : E F = C E $\div \frac{x}{y}$ $\therefore$ .065912 : .36108 : : 131.394 : E F = $\frac{131.394 \times .36108}{.065912} = 719.8 =$ E F.*

Then H F = E F − E H = 719.8 − 250 = 469.8.

Then H F = I K = 469.8.

If the rebutment-pier be a triangle, the angle I B A = I B K. To find I A. As rad. : B I : : co-tan. ∠DBC : I A $\therefore$ 1 : 115.697 : : 7.953 : 115.697 × 7.953 = 920 = I A.

Fig. 8. To find O K, I O (= L V = B S = $\frac{BW}{2}$), V O (= I B + B L) and V F (= the focal distance); the height B I, the direction B T, and P B the height through which a falling body will acquire the projectile velocity, being given.

Let B I = a
L P = V F = x
P B = t
The angle L B H = D B R = ϕ

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* 131.394 = 2.1185755 log.
.36108 = $\bar{1}$.5576034 log.
1.6761789 log.
.065912 = $\bar{2}$.8189679 log.
719.8 = 2.8572110 log.

Then $B\,L = t - x$ Fig. 8.

also $L\,H = \sqrt{P\,L.B\,L} = \sqrt{x\ (t-x)} = \sqrt{tx - x^2}$

also $L\,H : L\,B :: r : \text{tang.}\ \angle\ L\,H\,B = \phi$

$\sqrt{tx - x^2} : t - x : r : \text{tang.}\ \phi$

$x\ (t - x) : (t - x)^2 : r^2 : \text{tang.}^2\phi$

$x : t - x :: r^2 : \text{tang.}^2\phi$ or if $r = 1$

$x : t - x :: 1 : \text{tang.}^2\phi$

$t - x = x\ \text{tang.}^2\phi$

$t = x + x\,\text{tang.}^2\phi = (1 + \text{tang.}^2\phi)x = \text{sec.}^2\phi x$

$\therefore\ x = \frac{t}{\text{sec.}^2\phi} = V\,F = t\ \text{cos.}^2\phi.$

$\therefore\ B\,L = t - x = x\ \text{tang.}^2\phi = \frac{t}{\text{sec.}^2\phi} \times \text{tang.}^2\phi = \frac{t.\ \text{tang.}^2\phi}{\text{sec.}^2\phi} = t\ \text{sin.}^2\phi$ that is $B\,L = t\ \text{sin.}^2\phi.$

$L\,H = \sqrt{P\,L\ .\ B\,L} = \sqrt{t\ \text{cos.}^2\phi\ .\ \text{sin.}^2\phi}$
$= t\ \text{sin.}\phi.\text{cos.}\phi.$

$I\,O = L\,V = 2\ L\,H = 2\ t\ \text{sin.}\phi.\ \text{cos.}\phi.$

$B\,W = 4\ L\,H = 4\ t\ \text{sin.}\phi.\ \text{cos.}\phi.$

$L\,I = I\,B + B\,L = a + t\ \text{sin.}^2\phi = V\,O$

$O\,K = \sqrt{4.V\,F.V\,O} = 2\sqrt{V\,F.V\,O}$
$= 2\sqrt{t\ \text{cos.}^2\phi\ (a + t\ \text{sin.}^2\phi)}$

$I\,K = I\,O + O\,K = 2\ t\ \text{sin.}\phi\ .\ \text{cos.}\phi +$
$2\ \sqrt{t.\text{cos.}^2\phi\ (a + t\ \text{sin.}^2\phi)}.$

To find $t = P\,B$ supposing $I\,K$ given.
Let $I\,K = 2\ b.$

Then

Fig. 8. $2\,b = 2\,t\,\sin.\phi.\cos.\phi + 2\sqrt{t\cos.^2\phi\,(a + t\sin.^2\phi)}$

$b - t\sin.\phi.\cos.\phi = \sqrt{t\cos.^2\phi\,(a + t\sin.^2\phi)}$

$b^2 - 2\,b\,t\sin.\phi.\cos.\phi + t^2\sin.^2\phi.\cos.^2\phi)$

$= a\,t\cos.^2\phi + t^2\sin.^2\phi.\cos.^2\phi$

$b^2 = a\,t\cos.^2\phi + 2\,b\,t\sin.\phi.\cos.\phi$

$= \cos.\phi\,(a\cos.\phi + 2\,b\sin.\phi)t$

$$\therefore t = \frac{b^2}{\cos.\phi\,(a\cos.\phi + 2\,b\sin.\phi)} = 321.194$$

When K I $= 2\,b = 469.8$ then $b = 234.9$

B I $= a = 115.697$ and $\phi = 7°\,10'$

B W $= 4t\sin.\phi.\cos.\phi = 159.031$ and B L

$= t\sin.^2\phi = 4.999$

B I + BL = V O = 115.697 + 4.999 = 120.696.*

---

* To find P B or the velocity at B corresponding to the range on the inclined plane B K.

T B W $= \phi = 7°\,10'$

K I $= 2\,b = 469.8$ then $b = 234.$

B I $= a = 115.697$

$$\text{P B} = t = \frac{b^2}{\cos.\phi\,(a\cos.\phi + 2\,b\sin.\phi)}$$

Log. of $a = 2.0623221$

Log. of cos. $\phi = \bar{1}.9965937$

$2.0589158 = 114.529$

To find the absciss, V O, and double ordinate B W of the catenary F *w* *v* B

Fig. 4.

$$\begin{array}{rl}
 & \text{Brought up } 114.529 \\
2b=469.8 \text{ its log.} & =2.6719130 \\
\text{Log of sin.}\varphi & =\bar{1}.0960615 \\
 & \underline{1.7679745} = \underline{58.614} \\
 & \phantom{1.7679745 = {}} 173.143
\end{array}$$

$$\begin{array}{rl}
\text{Log. of cos. }\phi & =\bar{1}.9965937 \\
\text{of } 173.143 & =2.2384049 \\
 & \overline{2.2349986} \\
\text{Log } b & =2.3708830 \\
 & 2 \\
\text{of } b^2 & =\overline{4.7417660} \\
 & 2.2349986 \\
 & \overline{2.5067674}=321.194=t=\text{PB.}
\end{array}$$

To find $\text{B W}=4t \text{ sin.}\varphi.\text{cos.}\varphi$

$$\begin{array}{rl}
\text{Log. } 4 & =0.6020600 \\
t & =2.5067674 \\
\text{Sin. }\varphi & =\bar{1}.0960615 \\
\text{Cos. }\varphi & =\bar{1}.9965937 \\
 & \overline{2.2014826} = \underline{159.031 = \text{B W.}}
\end{array}$$

To find $\text{B L}=t \text{ sin.}\phi^2$.

$$\begin{array}{rl}
\text{Log. sin.}\phi & =\bar{1}.0960615 \\
 & \phantom{=\bar{1}.096061}2 \\
\text{of sin. }\varphi^2 & =\bar{2}.1921230 \\
t & =2.5067674 \\
 & \overline{.6988904}=4.9991=\text{B L.}
\end{array}$$

$$\begin{array}{rl}
 & \text{B I}+\text{B L}=\text{VO} \\
\text{B I} & =115.697 \\
\text{B L} & =4.999 \\
\text{V O} & =\overline{120.696}
\end{array}$$

Fig. 4. assumed by the chain F B being let down from F, so that the part of the curve, $B\,v\,w$, may be similar to the catenary B C G, that is, the angle $w\,B\,v =$ G B C.

Put the tension at $V = c$.

The tabular value corresponding to the $\angle N\,w\,v$ from column $x = x$.

That corresponding to $n\,w = y$.

$B\,H = O\,n = a$.

$H\,F = b$.

And the $\angle F = \phi$.

Then

$v\,n = c\,x$, and $n\,w = c\,y$.

Consequently

$v\,O = O\,n + v\,n = a + c\,x$.

$O\,F = H\,F - H\,O = H\,F - B\,n$

$= H\,F - n\,w = b - c\,y$.

But, by the equations of the catenary, $v\,O = c\,(\sec.\phi - 1)$ and $O\,F = cm.\log.\,\text{tang.}\,(45^\circ + \frac{1}{2}\,\phi.)$

Whence

$c\,(\sec.\,\phi - 1) = a + c\,x$.

or $c\,(\text{sec.}\ \varphi - 1 - x) = a.$ Fig. 4.

$$c = \frac{a}{\text{Sec.}\ \varphi - 1 - x}$$

And

$$c\,m\ \text{log. tang.}\ (45^\circ + \tfrac{1}{2}\,\varphi) = b - c\,y$$

$$c\,[m\ \text{log. tang.}\ (45^\circ + \tfrac{1}{2}\,\varphi) + y] = b$$

$$c = \frac{b}{m\ \text{log. tang.}\ (45^\circ + \frac{1}{2}\,\varphi) + y}$$

Consequently

$$\frac{a}{\text{Sec.}\ \varphi - 1 - x} = \frac{b}{m\ \text{log. tang.}\ (45^\circ + \frac{1}{2}\,\varphi) + y}$$

Or

$\dfrac{m\ \text{log tang.}\ (45^\circ + \frac{1}{2}\,\varphi) + y}{\text{sec.}\ \varphi - 1 - x} = \dfrac{b}{a}$ an equation, in which φ is the only unknown quantity; and hence the value of φ may be easily found by approximation.

It may be further remarked, on this equation, that the value of m log. tang. $(45^\circ + \frac{1}{2}\,\varphi)$ is the number in column y, in the table opposite the required $\angle\ \varphi$, and the value of sec.$\varphi - 1$ is the number in column x, opposite the same angle.

We may likewise transform the foregoing equation into the following, viz.

$$\frac{\text{Log. tang.}\ (45^\circ + \frac{1}{2}\,\varphi) + \frac{y}{m}}{\text{sec.}\ \varphi - 1 - x} = \frac{b}{am}$$

Fig. 4. Or, if we call the given angle $n\,w\,v = \varphi'$, this equation may again be transformed into

$$\frac{\text{Log. tang. }(45^\circ + \frac{1}{2}\varphi) + \text{log. tang. }(45^\circ + \frac{1}{2}\varphi')}{\text{sec. }\varphi - \text{sec. }\varphi'} = \frac{b}{am}$$

an equation by which φ may be found, by an easy approximation, without the assistance of the catenarian table.

EXAMPLE.

$$\frac{\text{Log. tang. }(45^\circ + \frac{1}{2}\varphi) + \text{log. tang. }(45^\circ + \frac{1}{2}\varphi')}{\text{sec. }\varphi - \text{sec. }\varphi'} = \frac{b}{a\,m}$$

$$B\,H = a = 115.69$$
$$H\,F = b = 469.8$$
$$n\,w\,v = \varphi' = 7^\circ,\!,10'$$
$$m = 2.302585$$

Then $\dfrac{b}{a\,m} = \dfrac{469.8}{115.69 \times 2.302585} = 1.763$

Log. of 469.8 = 2.6719130

Log. of 115.69 = 2.0632958

Log. of 2.302585 = 0.3622156   2.4255114

.2464 01 = 1.763

Log. tang. $(45^\circ + \frac{1}{2}\varphi')$ = log. tang. $48^\circ,\!,35'$ = .0544646

And sec. φ' = sec. $7^\circ,\!,10'$ = 1.007874

Hence,

$$\frac{\text{Log. tang. }(45^\circ + \frac{1}{2}\varphi) + .0544646}{\text{Sec. }\varphi - 1.007874} = 1.763$$

Suppose $\varphi = 32^\circ$. Then $45^\circ + \frac{1}{2}\varphi = 61^\circ$

And $\frac{.2562480\ +\ .0544646}{.179178\ -\ 1.007874} = \frac{.3107126}{.1713} = 1.8138$ Fig. 4.

which ought to be 1.763.

$$\begin{aligned} &\text{Log. of } .31071 = \bar{1}.4923552 \\ &\text{Log. of } \ .1713 = \underline{\bar{1}.2337574} \\ &\text{Log. of } 1.8138 = \ \ .2585978 \end{aligned}$$

Suppose $\varphi=33°$. Then $45°+\frac{1}{2}\varphi=61°{}_{\prime\prime}30'$.

And $\frac{.2652356\ +\ .0544646}{1.192363 - 1.007874} = \frac{.3197}{.1845} = 1.7328$

which ought to be 1.763.

$$\begin{aligned} &\text{Log. of } .3179 = \bar{1}.5047426 \\ &\text{Log. of } .1845 = \underline{\bar{1}.2659964} \\ &\text{Log. of } 1.7328 = \ \ .2387462 \end{aligned}$$

As diff.

$$\begin{array}{lllll} 1.8138 & 33° & & 1.763 & \\ \underline{1.7328} & \underline{32°} & & \underline{1.733} & \\ .081 & : 1° & :: & .03 & : \end{array} \frac{1° \times .03}{.081} = \frac{60' \times .03}{.081} = 22'$$

Therefore $\varphi = 32°\ 22'$ which lies between $32°{}_{\prime\prime}4'$ and $32°{}_{\prime\prime}50'$ in the tables.

By the formula, m log. tang. $(45°+\frac{1}{2}\varphi)=$ tab. y, and $\sec.\varphi-1=$tab. x.

Then at the angle $32°\ 22'$ tab. $y = .5974$, and tab. $x=.184$

and $\frac{m.\text{log.tang.}(45°+\frac{1}{2}\phi)+y}{\sec\ \phi-1-x} = \frac{.5974+y}{.184-x} = \frac{.5974+.1254}{.184-.00787} = \frac{.722}{.176}$ $=4.$ and $\frac{a}{b}=\frac{469.8}{115.69} = 4$, which verifies the former operation.

Fig. 4. Having found the angle,

Then, by reference to the table, at the angles 7°,10′ and 32°,22′,

$$\text{Tab.}\,Ov-\text{tab.}\,nV=\text{tab.}\,BH$$

$$.184-.00787=.1761$$

$$\text{Tab.}\,Bn+\text{tab.}\,OF=\text{tab.}\,HF$$

$$.1254+.5974=.7228$$

As tab. B H : tab. $n\,v$: : B H : required $n\,v$

$.1761 : .00789 : : 115.69 : \frac{115.69\times .00789}{.1761}=5.17$

Then

$VO=BH+n\,v$ Log. of $115.69=2.0632958$

$VO=115.69+5.17$ of $.00787=\bar{3}.8959748$

$VO=120.86$ $\bar{1}.9592706$

$.1761$ $\bar{1}.2457594$

$5.17=.7135112$

As tab. HF : tab. Bw : : H F : required B w

or $.7228 : .2508 : : 469.8 : \frac{469.8\times .2508}{.7228}=163.1$

Log. of $469.8=2.6719130$

of $.2508=\bar{1}.3993275$

2.0712405

$.7228=\bar{1}.8590181$

163.1 2.2122224

Hence it appears, that whether the extrados of the pier be a catenarian curve or a parabolic, the difference will be very trifling.

By the parabolic, B w = 159
By the catenarian, B w = 163
By the parabolic, v O = 120.69
By the catenarian, v O = 120.86

Let I B W K be a bearing pier, and the triangle B I A similar to the triangle B D L, B D and B I homologous sides. Fig. 9.

It is manifest that the smaller the angle (D B C = D B L) the catenary makes with a horizontal line, the greater will be the lateral force represented by B I, and the smaller the bearing force represented by I A. And as the pier B I is higher or lower, these measures will increase or decrease.

Let the bridge, including its incumbent weight, weigh 1175 tons, as before; and let a foot cube of the stone of which the pier is to be built weigh 136 lbs. If the bridge be 30 feet wide,

$$\frac{1175 \times 2240 \text{ (lbs. in a ton)}}{30 \times 136} = 645$$

Then, if the pier be one foot wide it would have to bear a weight equal to a column of the same stone, one foot square in section, and 645 feet high.

Perronet (page 626, des Ponts,) says, to crush a foot cube of the stone of Saillancourt, (weighing per foot cube French, 152 French pounds *,) of which the bridge of Neuilly is built, of the same specific gravity as that taken above, required, a column of the same base 1580 feet high; but not more than one-twelfth of that weight was laid upon it in any part.

Suppose the stone of this bridge to come from Saillancourt, then $\frac{1580}{12} = 132$, the practical strength of the stone; therefore $\frac{645}{132}$ will be the width of the pier, say 5 feet. Hence if the pier be no wider than 5 feet, it would have the practical strength of the bridge of Neuilly, in respect to its liability of being crushed. But the pier is 116 feet high.

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* A French foot cube of Saillancourt stone weighs 152 French pounds, therefore an English foot cube will weigh 136 English pounds of that stone.

Let the angle DBC, which the catenary makes with a horizontal line be 7°.10′, and B I (= 116 feet,) the height of the pier at B. Draw B A at right angles to the tangent of the catenary at B, intersecting the base of the pier in A. Make the width of the pier = 2 A I; then, as rad. : tang. $\angle 7^\circ\ 10''$ :: 116 : 14.5, hence 2 A I =29 feet, which is about the fourth of its height; which, with reference to practice, seems a proper proportion in this case. Fig. 9.

Let the length of the ordinate D B of the catenary be 250 feet, then the length of the curve will be 250.655 feet; so that if the pier were B A W, and vibrated upon the point A, V A could not assume the place of A W, so long as the horizontal resistance remained constant. The parabolic curve, B V W, will be obtained by the example before given, B W being the same in this as I K in that; B W being the greatest range of a projected body on a horizontal plane, B W, instead of the greatest range on the inclined plane B K.*

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* The catenarian B W may be obtained easily from the

It is usual to make columns and walls diminish in thickness as they increase in height; but this case should be an exception; for it is important to reduce the ordinate of the catenary as much as possible, because the diminution of the ordinate, and, consequently, the length of the chain and tension, (the angle being the same,) more than counterbalances the excess of stone required in a pier whose section is a parallelogram above one whose section is the frustrum of a triangle. Therefore the pier should have perpendicular sides, B I . W K, instead of battering sides O E, as in the proposed bridge over the Strait of Menai.

In the bridge of Salamanca, the bearing

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tables, being a similar curve to that of the catenary of the bridge, viz. at the angle 7° 10′

Then as tab. $2y$: tab. x : : 2 A I : V N

$$.2508 : .00787 :: 29 : \frac{29 \times .00787}{.2508} = \frac{1.4623980 \text{ log.}}{1.3993275 \text{ log.}}$$

$$.0630705 \text{ log.}$$

$$= 1.1563 = \text{V N}$$

making the pier at V a little more than one foot higher than at B.

piers are 155 feet high and 27 feet wide, $\frac{27}{155} = \frac{1}{5.7}$

In the bridge near Terni, the bearing piers are 108 feet high and 27 feet wide, $\frac{27}{108} = \frac{1}{4}$

In the aqueduct of Spoletto the bearing piers are 260 feet high and 11.7 wide, $\frac{11.7}{260} = \frac{1}{22}$

The above piers are in section parallelograms.

In the aqueduct at Lisbon, the bearing piers batter; their height is 150 feet, their width at bottom 25 feet, then $\frac{25}{150} = \frac{1}{6}$, the difference in width between the bottoms and tops of these piers is not much. (See Gauthey, des Ponts.)

1st. It appears manifest that there is no danger of the stone being crushed if the pier be wider than five feet at the top, and in a bearing pier the horizontal draft is supposed nothing; therefore that would be a sufficient width to prevent the stone being crushed, supposing it of no height, by calculation.

But, with reference to a pier of the prodigious height of 116 feet, one-fourth the height, or 29 feet, is found a proper width.

In the practical cases referred to, there is comparatively no vibration; and therefore that width in those cases might be ample upon the ordinary mode of construction.

The Edystone lighthouse, from the ground course of the rock to the summit of the light, is 93 feet; its width, at that course, 32 feet; and, at the top, under the cornice, 15 feet; it is constructed throughout with vertical bond. The vibration of a chain-bridge whose distance between the points of suspension is 500 feet, may be compared with that of the force of the wind and sea in respect to Edystone light-house, to separate the stones from each other; and, therefore, if the pier be in width of a proportion to its height, warranted by other cases nearly approaching to it, then, in its construction, it ought also to have reference to the case nearest approaching to it with regard to the force which it may have to resist.

Hence, such a pier ought to be constructed with courses having a vertical bond, either by dovetailing, joggling, or by laying the courses so as to produce one solid block. But if the pier be built in the ordinary manner, without vertical bond, then it may, in the words of Mr. Emerson, be said to be built of loose materials; and then, in such a case, it would be better that the bearing pier should be 159 feet wide, or 163 feet, as obtained from the parabola or catenary.

But in no case need the pier be solid; it may be hollow, perpendicularly or horizontally, it may have wells in it or arches through it.

If the horizontal draft be resisted by the land rebutments wholly, that is, if the bearing piers be considered wholly bearing piers, then the tops of the bearing piers should be made upon the principle of a railway, so that there may be as little friction between the chains and the bed on which they lie as possible; perhaps the chains should lie on rollers.

GEOMETRICAL METHODS OF DETERMINING THE RELATIONS BETWEEN THE PARTS OF A CATENARY.

THE following is a translation of John Bernouilli's introduction to his Lectio de Curvis Furniculariis vel Catenariis: —

"Of what great use the problem of the catenarian line may be in geometry, the three solutions inserted in the Act. Leipsien. 1691, and especially those of the celebrated Leibnitz will show. Galileo was the first who thought of the curve formed by a thread, or rather a small chain inextensible and hanging freely; but he did not penetrate its nature, for he determined it to be a parabola, which in no wise it is.

"J. Jungius, as Leibnitz has mentioned, by calculation and many experiments, found that it is not the parabola, but did not ascertain the true curve. The solution of this very excellent problem was reserved for our time.

"The curve of the catenary is of two kinds;

either the common, which is formed by a thread or chain equally thick, or equally heavy in all its points; or uncommon, which is formed by a thread unequally thick, that is, which in all its points is unequally heavy, and in some ratio of the ordinates of a given curve." (See page 491, omnia opera.)

To draw the common catenary mechanically. Fig. 10.

Suspend on a vertical plane a chain of similar and equal links of homogeneous matter, as flexible as possible, from any two points not in a perpendicular line, nor so distant from each other as the length of the chain. Prick the plane through the links as nearly as possible in the middle of the chain, and through the points draw the catenary.

If the chord F B D or F *b d* be given, and the absciss B A or *b* A intersecting it in B or *b* at a given angle. Draw the vertical line B A, and F B D or F *b d* at the given angle on the plane. Fix one end of the chain at F, and from the point D or *d*, with another part of the chain, raise or fall the chain until the lowest part coincides with A, and through points made as before draw the curve.

Fig. 10. To draw a tangent to the catenary. Let D B E be a horizontal line, and at right angles to B A from A draw A R equal to the curve D A obtained as before, and draw B R, which bisect in *o*. At right angles to B R draw *o* C, intersecting B A continued in C. Draw C R, and make the angle B D T equal to the angle A C R. D T is the tangent required, and B C equals C R; C A is the tension at the point A, or the horizontal draft, which in a catenary is in every point the same, and is called the constant quantity; as D B : B T : : C A : A R; or as D B : B T :: the constant quantity C A to A R equal to the length of the chain, A D.

If C H be drawn through C at right angles to B C, it is called the directrix, and D H drawn parallel to B C, intersecting the directrix at H, is the tension at the point D, being always equal to the sum of the absciss and constant quantity. With the centre C and radius = the tension D H at D = C B, cut the tangent at the vertex A in R, then A R is the length of the chain A D.

A C is the semi-axis of an equilateral hyperbola, and also the radius of curvature of a circle equicurved with it and the catenary.

In the triangle C A R, when C A is the radius, then the tension equals C R, the secant of the angle A C R (= B D C.) The chain A D equals A R, the tangent of the same angle, and the absciss A B equals C R — C A = S R. Hence A C R being a right-angled triangle, it is manifest, that when two of the five quantities, viz. the angle, the absciss, the length of the chain between the vertex and point of suspension, the constant quantity or tension at the vertex, and the tension at the points of suspension, are known, the other three may be obtained geometrically, or from a table of tangents and secants.

Fig. 10. When the tension at the point of suspension is a minimum in respect to the ordinate, the ordinate is a fourth proportional to the length of the semichain, the tension at the vertex and that at the point of suspension, the angle made by the catenary with a horizontal line is 56°.28′, and the tangent D T passes through the axis continued at a distance A C from the apex, equal to the constant quantity, or where the directrix cuts the prolongation of the axis. In the right-angled triangle V C R, let the angle V C R be 56°.28′, with the centre C and radius C R describe the circle R B cutting C V continued in B. Through B, at right angles to B C, draw D F; and from C, C F at right angles to C R, cutting D F in F, and V R continued in E, make B D equal B F. F and D are points in a catenary.

With C as a centre and radius C V cut C R in S, and join D C.

Then in the right-angled triangle E C R all the quantities of the catenary D V F, whereof the angle B D C = 56°.28′ = V C R are represented, V C being a perpendicular let fall from C on the hypothenuse E R of the triangle.

V R = the length D V.

C S = V C = the constant quantity and radius of curvature at the vertex.

E C = D B = the ordinate.

C R = C B = the tension and subtangent at D or F.

S R = B V = the absciss.

B D C = V C R = the angle = R E C.

And C F = E R = the tangent.

By referring to a table of natural cosecants it will be seen that the natural cosecant C E of the angle 56°.28′ is 1.2 nearly, therefore, in that case, the constant quantity is $\frac{5}{6}$ of the ordinate whence the construction of the right-angled triangle without a scale of chords.

Let C Y be the directrix, and C V the constant quantity, when C F touches the curve, *a* V *t*, then the tension F L, with respect to the ordinate F B, is a minimum. But if C F cut the curve it must cut it in two points, as A *a*. If a catenary *a v t* be made similar to A V T to the same ordinate *a z*, Fig. 12.

the two catenaries *a v t*, *a* V *t* will have the same tension *a* Y at the points of suspension; that is, inasmuch as the absciss *z v* is shorter than *z* V, so much is the tension *v* C at the vertex longer. Hence, (except when the angle made by the catenary with a horizontal line be 56°„28′,) the ordinate and tensions at the point of suspension being given to draw the catenary, two catenaries may be obtained, one above and one below that angle.

Fig. 13. To draw the common catenary the tension at the apex being given.

From the indefinite right line E R, let fall the perpendicular V C, equal the tension at the apex. With C as a centre and radius, C E equal $\mathrm{C\,V} + \frac{\mathrm{C\,V}\,*}{5}$ cut E R in E, join C E. From C draw C R at right angles to C E, cutting E R in R. With C as a centre and radius C R cut C V continued in B. Through B draw F D, and make F B = B D = E C; V, D, and F are points in the catenary. Through C parallel to E R draw the directrix Y A, and parallel to B C, draw F L and D K, cutting the directrix in L and K. Make D U in K D continued, and F G in F L, each equal V R. Bisect C L, and from M the point of bisection draw M J perpendicular to C L. In B C continued make C *d* equal G L, and on V *d* as a diameter, describe the arc of a circle *a d*, cutting C L in *a*. Make M J equal *a c*, therefore a mean proportional to C V and G L. In like manner bisect C K in N, and make N O a mean proportional to C V and K U. Join O J, cutting B V in G. Through G draw I S, cutting M J continued, and N O in I and S, two other points in the catenary; hence by mean proportionals any number of points in the catenary F V D may be found. In order to extend the curve, make Y L in the directrix continued equal L M, then M J will be an extreme and G L a mean to W Y the other extreme drawn parallel to C B obtained by making M *e* on the directrix equal to G L; and with a centre in M J describe a circle through

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* V C R may be made an angle of 56°„28′ for the co-secant E C of that angle is one-fifth longer than the radius A C nearly.

J and *e*, and cutting J M continued in *i* make Y W = M *i*. In like manner an extreme to the point A, taken at a distance, K A equal K N may be obtained; and the line joining their extremities will cut the axis continued in Z, whence, as before, the ordinates Z Q and Z T. In like manner any number of points in the catenary may be obtained. If a curve be drawn through the points U O V J, it is a logarithmic curve whose subtangent is C V, and axis Y A.

With the centre C and radius C Z cut V R continued in R′. Draw R *h*, R′ H parallel to C Z, cutting the ordinates Z T, B D continued in *h*, H, points in an equilateral hyperbola whose semi-axis equals the tension of the catenary at the apex and ordinates, B *h*, Z H equal the length of the catenarian curves D V, T V.

If *f* and X, two points in the axis, continued, be taken each at a distance from V equal to 2 V C and through X a line be drawn parallel to V R, and points P *p* be taken in the ordinates of the catenary continued equally distant from the point *f* and line X *x* *x*′, P *p* will be in a parabola whose focus is in the point *f*, whose directrix is the line X x x'. and the length of whose curve p V or P V is equal to F h or Q H, that is, equal to the curve of the catenary V D (= B h, the ordinate of the hyperbola,) plus the ordinate of the catenary.

To draw the common catenary from an equilateral hyperbola A H and a parabola A P, whose parameter is equal to four times the axis of the hyperbola. Fig. 14.

Continue an ordinate H B of the hyperbola and make H F equal to the parabolic curve A P, F is a point in the catenary *

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* To describe a rectangular or equilateral hyperbola, viz. an hyperbola whose axes are equal, or whose assymptotes make a right angle, Fig. 31.

Let A L be the transverse diameter or axis and A the vertex of the hyperbola, and M N the conjugate diameter equal to A L.

From any point O in the conjugate diameter continued, draw O H parallel to A L and equal to O A. H is a point in the hyperbola.

Or,

Continue L A to B, and draw A G perpendicular to L A. From C the centre of the axis and any radius describe the arc B G. Draw G H parallel to A B and B H

B F = B D, the ordinate of the catenary, is equal to the parabolic curve A P less B H

parallel to G A. H where these lines intersect is a point in the hyperbola.

To describe a parabola whose parameter is equal to four times the axis of the foregoing hyperbola.

Draw AE a tangent to the vertex A. At right angles to A E draw A B and make A K equal to the parameter, equal to 4 A L. From any point N as a centre and radius N K longer than half the parameter, cut A B in B and AE in E. Draw B P parallel to A E and E P parallel to A B intersecting in P a point in the parabola. From any other point *n* and radius *n* K cut A B in X, and proceed as before. T is a point in the parabola.

Or,

From a property of the parabola, the focal distance equals one fourth of its parameter, and by the proposition equal to A L. Make A V equal to A L, and draw L Y at right angles to AL. V is the focus and L Y the directrix of the parabola. Find any centres T and P of circles which shall touch the line L Y and pass through the point V. T and P, the centres, are points in the parabola.

Or,

From the focus V draw any line V Z intersecting A E in Z, and draw Z T at right angles to V Z. Make Z S equal to Z A, and let fall the perpendicular S T intersecting Z T in T, a point in the parabola.

the correspondent ordinate of the conterminate hyperbola A H.

The parabolic curve A P = H B + B F.

The curve of the catenary A D is equal to B H the corresponding ordinate of the conterminate hyperbola.

To find the curve B X by the evolution of which the common catenary D A F is described, whose tension at the vertex is A C. Fig. 15.

Produce B D. Then A C : C B : : B H the ordinate of the hyperbola or the curve A F : F Z. Draw the perpendicular Z X and make it equal to B C. X is a point in the curve. The other points may be found in the same manner. Hence, to find the catenary, from the point X as a centre and radius equal to the curve X B + A C = X F intersect B H continued in F, which is a point in the catenary, and X F is the radius of curvature to the point F, or of a circle equicurved with the catenary at F, and F T at right angles to X F is the tangent at the point F. The evolving right line X F is a third proportional to A C and C B.

A C is the radius of curvature of the

catenary at the vertex, or of a circle equicurved with the chain there, and of the equilateral conterminate hyperbola. The curve B X is a third proportional to the right line A C and the curve A D *, B I is

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* The following are the common methods of finding proportionals.

Fig. 16. To find a third proportional to two given lines A B . B C.

Draw any angle, and make A B equal to A B, and B C and A D each equal to B C. Join B D and draw C E parallel to B D, intersecting A D continued in E. D E is the third proportional required, as A B : A D : : B C : D E.

Fig. 17. To find a mean proportional between two given lines A B . B C.

In any line make A B equal to A B and B C equal to B C, and bisect A C. With the point of bisection E, describe the semicircle A D C, and draw B D at right angles to A C, cutting the circumference in D. B D is the mean proportional required, as A B : B D : : B D : B C.

Having the sum of the two extremes A C and the mean D B, to find the extremes.

From the point E raise E *d* at right angles, equal to the mean, and draw *d* D parallel to A C, cutting the circumference in D, let fall D B perpendicular to A C, A B. B C are the extremes.

double A B. Draw H R and F G parallel o A B, cutting R G the tangent of the catenary at the vertex A. $AC \times DH =$ the area A G F and $AC \times BZ =$ the area $ABHR + AGF =$ the area 2 B A H. $FAD = \overline{FD \times BA} - \overline{2AC \times DH}$.

To draw the common catenary from the logarithmic curve, let Y C K be an indefinite horizontal line, and from it describe a logarithmic curve. Fig. 19.

Let the ordinate A C be selected equal to its subtangent Y S, * which, in this

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To find a fourth proportional to A C, B C, A F. Fig. 18.

Draw any angle, and make A C, B C and A F respectively equal to the three given lines A C, B C, A F. Join C F and draw B Z parallel to C F, intersecting A F continued in Z. F Z is the fourth proportional required, as A C : C B : : A F : F Z.

* To draw a logarithmic curve with the given subtangent Y S from the hyperbola and parabola. Draw the horizontal line Y C K. From any point C let fall the perpendicular C A equal Y S. Through A draw R A parallel to Y C K. At any angle with C A draw C R, cutting A R in R. With C as a centre and Fig. 19.

curve, is every where equal or constant, and two points, Y and K, be taken in

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radius C R, cut C A continued in B, and through B draw F P parallel to A R. Make B H in B P equal R A. Draw the parabola A P whose vertex is A, and focal distance equal 2 C A and make H F equal to the parabolic curve P A. Make B D equal B F, and draw K F and Y D parallel to B C, cutting Y C K in Y and K. Then K F + Y D equals the sum of the extremes to C A the mean. With the centre R and radius R A, cut R C in o, o C is one extreme. In K F make K G equal C o, and through B draw the right line G B L, cutting Y L, continued in L. Y L is the other extreme, G A L are points in the logarithmic curve. * Bisect K C in c, continue A C, and make C e = K G. With the diameter A e describe the semicircle A i e, cutting C K in i. From c in C K draw c d perpendicular to C K and equal to C i, d is another point in the curve. In like manner a mean proportional to C A and Y L may be found; and by bisecting the distances, and from the extremes, any mean, and therefore any intermediate point, may be found. To find a point V more distant from A than G, make B A equal A C and B v equal K G and with x in A B, as a centre describe the circle

* Or from B, as a centre and radius B C, describe the semicircle C M. Take the distance B a equal A C from the diameter C M, and draw a b parallel to C M, cutting the circumference in b. From b draw b Q perpendicular to a b, cutting the diameter in Q. Then Q M is one extreme, and C Q the other, and Q b the mean equal to C A.

Y C K, equally distant from C; from which draw two other ordinates Y L, K G. Let Y D in Y L and K F in K G continued each = the half the sum of Y L and K G; D and F are points in the catenary.

The space A C Y D is equal to the rectangle of C A and A R, and A *g* D equals a triangle under C A and *g* R.

Frezier, in his Traité de Stereotomie, vol. iii. page 369, has the following method (as he says) of drawing a catenary; and Rondelet, vol. ii. page 140., has copied it from him.

Let A M, M B, be the two ordinates to the catenary, and M S the absciss, to construct the curve A S B. With the centre M and radius M A describe the arc A C, and from C, with the same radius, describe the arc M E, cutting S D, a line drawn through Fig. 20.

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through *v* and A cutting A B continued in *w*, then B *w* is one extreme, B A the other, and K G the mean. Continue Y K, and make K W equal K C, and let fall from W, W V, perpendicular and equal B *w*. V is a point in the logarithmic curve.

the point S parallel to A B in the point E.

Continue M S, and make S P equal to S E.

Through P draw H I parallel to A B. From the points A and B erect perpendiculars A H, B I, cutting H I in H and I. With P, as a center, and radius P M, describe an arc, cutting S D in D and join D P. From D, with radius D S, cut D P in N, and make I L equal to N P.

Continue H A, and make H R a third proportional to I L, P S. Bisect H P, P I in G and K, and make G F a mean proportional to P S, H R, and K O a mean between I L and P S.

Through the points L O S F R draw the logarithmic curve.

Frezier and Rondelet say, if K V be made equal to the half sum of K O and G F, V will be a point in a catenary.

It is manifest that this method cannot be used to draw the curve said to be a catenary, when M B is less than half M S, and that S P is a variable quantity, and unless S P happens to be the subtangent of the logarithmic curve so drawn, it will not be the common catenary.

All catenaries making equal angles with their ordinates at the points of suspension are similar; therefore from a given catenary another similar may be drawn, as follows:

Let it be required to draw a catenary to the absciss *a b* and ordinate *a d*. Fig. 22. & 23.

Join *d b*, and from the vertex of another given catenary draw B D making the angle D B A = *d b a* and cutting the curve in D, whence by scales of equal parts the catenary *d b* may be drawn.

EXAMPLE.

Divide D B into any number of parts, and erect the perpendiculars EF, and C G, cutting the catenary in F and G. Join F B, G B. Divide *d b* (*fig.* 21.) into the same number of parts, and make the triangles *b e f*, *b c g* respectively similar to the triangles B E F, B C G, *f* and *g* are points in the catenary.

TO DETERMINE THE NUMBER OF SQUARE INCHES IN THE COLLECTIVE SECTIONS OF CHAINS SUPPORTING A ROAD.

SUPPOSE a road to be supported by chains suspended in the form of a catenary, and made of rope, wood, iron, steel, or any other materials; and that the roadway be either on or below the chains; but so arranged as not to alter the catenarian form of them: It is required to determine the number of square inches in the collective sections of the chains, when the weight of the road, the materials of which the chains are made, and any two of the five quantities φ, c, x, z or y are given.

Put n for the number of square inches in the collective sections of the chains.

g for the weight of a part of the same materials measuring *one inch* square by *one foot* long.

f for the weight or tension, which would just overcome the force of cohesion in a like piece of the same materials.

w the weight of the road upon each foot of the chains.

Then since g is the weight of a part of the chains measuring one inch square by one foot long, and n is the number of square inches in the collective sections of the whole, it follows that ng must be the weight per foot of all the chains; and consequently $ng + w =$ the weight per linear foot of the whole, chains and road-way, conceived as one chain. Hence by the subsequent investigation $(ng + w)\,t =$ the absolute tension at each point of suspension.

Again, since f is the force of cohesion in each square inch of the given materials, and n is the number of square inches in the collective sections of all the chains, nf must represent the tension which they will just bear.

Hence $nf = (ng + w)\,t = t\,ng + t\,w$ or $nf - ing = tw$, from which we have $n = \frac{tw}{f - tg} = \frac{w}{\frac{f}{t} - g}$ from which again by substituting for t its values found by the table, in terms of any two of the five quantities φ, c, x, z, and y, we have,

1. When c and φ are given $n = \dfrac{w}{\dfrac{f}{c \text{ sec.}\varphi} - g}$

2. When z and φ are given $n = \dfrac{w}{\dfrac{f}{z \text{ cosec.}\varphi} - g}$

3. When x and φ are given $n = \dfrac{w}{\dfrac{f \text{ ver. sin. } \varphi}{x} - g}$

4. When y and φ are given

$n = \dfrac{w}{\dfrac{f\,m}{y} \text{ cos.}\,\varphi \text{ log. tang. } (45^\circ + \frac{1}{2}\varphi) - g}$

5. When z and c are given $n = \dfrac{w}{\dfrac{f}{\sqrt{z^2 + c^2}} - g}$

6. When x and c are given $n = \dfrac{w}{\dfrac{f}{c + x} - g}$

7. When x and z are given $n = \dfrac{w}{\dfrac{2\,f\,x}{z^2 + x^2} - g}$

8. When c and y are given, find φ from the formula log. tang. $(45^\circ + \frac{1}{2}\varphi) = \frac{y}{c\,m}$. Then $n = \dfrac{w}{\dfrac{f}{c \text{ sec.}\varphi} - g}$

9. When z and y are given, find φ from the formula cot. φ log. tang. $(45^\circ + \frac{1}{2}\varphi) = \frac{y}{m\,z}$

Then $n = \dfrac{w}{\dfrac{f}{z \text{ cosec.}\varphi} - g}$

10. When x and y are given, find φ from the formula

$\frac{\text{log. tang.}(45^\circ+\frac{1}{2}\phi)}{\text{sec.}\phi.\ \text{ver. sin.}\phi}=\frac{y}{m\,x}.$ Then $n=\frac{w}{\frac{f\ \text{ver. sin.}\ \phi}{x}-g}$

Note. In every case when the first term in the denominator is either equal to, or less than g, the solution to the problem is impossible. It may also be observed, that the value of ϕ, in Cases IX. and X. may be found by approximation, as in the subsequent investigation.

Adopting the notation employed in the last article*; it is required to determine the catenary, when n, g, f, w, and any one of the five quantities ϕ, c, x, z, and y are given.

From the expression $(ng+w)\,t=nf$, derived in the last problem,

we have $t=\frac{nf}{ng+w}=\frac{f}{g+\frac{w}{n}}$

Hence by substituting the values of t found by the catenarian table, we have

1. When ϕ is given, $c.\text{sec.}\phi=\frac{f}{g+\frac{w}{n}}$ or $c=\frac{f\ \text{cos.}\phi}{g+\frac{w}{n}}$

2. When c is given, $\text{sec.}\phi=\frac{f}{c\left(g+\frac{w}{n}\right)}$

3. When x is given, $\frac{x}{\text{ver. sin.}\ \phi}=\frac{f}{g+\frac{w}{n}}$

or ver. sin.$\varphi = \dfrac{x\left(g+\frac{w}{n}\right)}{f}$

4. When z is given, z cosec. $\varphi = \dfrac{f}{g+\frac{w}{n}}$ or

cosec. $\varphi = \dfrac{f}{z\left(g+\frac{w}{n}\right)}$

5. When y is given, $\dfrac{y}{m \cos.\varphi \log.\text{tang.}(45^\circ+\frac{1}{2}\varphi)} = \dfrac{f}{g+\frac{w}{n}}$

or cos. φ. log.tang. $(45^\circ + \frac{1}{2}\varphi) = \dfrac{y\left(g+\frac{w}{n}\right)}{mf}$ from which φ may be found by approximation.

Corollary. From this problem we have an easy method of determining the greatest catenary which can be formed of any given materials, supposing any one of the five quantities φ, c, x, z, and y, to be given. For we have only to make w in the above formulæ $= o$, and n will then disappear; and consequently,

1. When φ is given, $c = \dfrac{f \cos.\varphi}{g}$

2. When c is given, sec.$\varphi = \dfrac{f}{cg}$

3. When x is given, ver. sin. $\varphi = \frac{gx}{f}$

4. When z is given, cosec. $\varphi = \frac{f}{zg}$

5. When y is given, cos. φ log. tang. $(45^\circ + \frac{1}{2}\varphi) = \frac{y\,g}{m\,f}$

USE OF THE TABLE OF THE DIMENSIONS OF THE CATENARY.

In order to make the calculation of the different parts of the catenary as easy as possible, the following table has been carefully constructed, making the tension at the vertex = 1. And, in order to illustrate its application, we shall, by its assistance, show how to solve the following cases.

CASE I.

When the angle at the point of suspension, and the tension at the vertex, are given, look in the column marked φ of the table, and take out x, y, z and t, standing opposite to φ in their respective columns. Then multiply each of them by the given tension c at the vertex, and the results will be, respectively, the absciss, ordinate, semi-curve, and tension at the point of suspension, of the proposed catenary.

CASE II.

When the angle at the point of suspension, and the tension at the same point are given,

Look in the table, in column marked φ for the given angle, and take out the opposite number from column t, then divide the given tension by that number, and the result will be the tension at the vertex of the proposed curve.

Again, from the columns marked x, y, and z, still opposite the given angle, take out the corresponding numbers, and multiply each of them by the tension at the vertex already found, and the results will be, respectively, the absciss, ordinate, and semicurve of the proposed catenary.

CASE III.

When the angle at the point of suspension, and the length of the semicurve are given,

Look in column φ for the given angle, as before, and take out the corresponding number from column z; then divide the given semicurve by that number, and the quotient will be the tension at the vertex of the proposed catenary.

Again, take out of the table the corresponding numbers from columns x, y, and t, and multiply each of them by the tension at the vertex already found, and the results will be, respectively, the absciss, ordinate, and tension at the point of suspension, of the proposed catenary.

CASE IV.

When the angle at the point of suspension and the absciss are given,

Look for the given angle in column φ, and take out the corresponding number from column x; divide the given absciss by that number, and the quotient will be the tension at the vertex of the proposed catenary.

Then take out the corresponding numbers from columns y, z, and t, and multiply each of them by the tension at the vertex already found, and the results will be, respectively, the ordinate, semicurve, and tension at the point of suspension, of the proposed catenary.

CASE V.

When the angle at the point of suspension and the ordinate are given,

Look in column φ for the given angle, and take out the corresponding number from column y. Divide the given ordinate by that number, and the result will be the tension at the vertex of the required catenary.

Then take out the corresponding numbers from columns x, z, and t, and multiply each of them by the tension at the vertex already found, and the results will be respectively the absciss, semicurve, and tension at the point of suspension of the proposed catenary.

CASE VI.

When the tension at the vertex and that at the point of suspension are given,

Divide the latter of these given quantities by the former, and find the result in column t of the table; then opposite, in column φ, will be found the angle at the point of suspension, with which, and the given tension at the vertex, the remaining quantities may be found, as in Case I.

CASE VII.

When the tension at the vertex and the semicurve are given,

Divide the latter of these given quantities by the former, and find the quotient in column z of the table; and opposite, in column φ, will be found the angle at the point of suspension, with which, and the given tension at the vertex, find the remaining quantities, as in Case I.

CASE VIII.

When the tension at the vertex and the absciss are given,

Divide the latter of these given quantities by the former, and find the quotient in column x of the table, and opposite, in column φ, will be found the angle at the point of suspension, by which, in conjunction with the given tension at the vertex, the remaining quantities may be found, as in Case I.

CASE IX.

When the tension at the vertex and the ordinate are given,

Divide the latter of these given quantities by the former, and find the quotient in column y of the table, and, opposite, in column ϕ, the angle at the point of suspension will be found; by which, in conjunction with the given tension at the vertex, the remaining quantities may be found as before directed.

CASE X.

When the semicurve, and the tension at the point of suspension are given,

Divide the latter of these given quantities by the former, and find the quotient in column $\frac{t}{z}$ of the table, and opposite will be found the angle at the point of suspension, by which, in conjunction with the semi-curve, the remaining quantities may be found, as in Case III.; or with the same angle and the tension at the point of suspension, the remaining quantities may be determined, as in Case II.

CASE XI.

When the absciss and tension at the point of suspension are given,

Subtract the former of these quantities from the latter, and the remainder will be the tension at the vertex of the proposed catenary, by which, in conjunction with either of the given quantities, the remaining ones may be found by Case VI. or VIII.

CASE XII.

When the ordinate and the tension at the point of suspension are given,

Divide the latter of these quantities by the former, and find the quotient in column $\frac{t}{y}$ of the table, then opposite, in column φ, will be found the angle at the point of suspension, by which, in conjunction with either of the given quantities, the remaining ones may be found, as in Case II. or Case V.

Note. If the quotient before mentioned be found in two places in column $\frac{t}{y}$ the problem admits of two solutions; and, if it cannot be found at all in that column, the solution to the problem is impossible.

CASE XIII.

When the absciss and the semi-curve are given,

Divide the latter by the former, and find the result in column $\frac{z}{x}$ of the table, and, opposite, in column φ, will be found the angle at the point of suspension, by which, in conjunction with either of the given quantities, the remaining ones may be found by Case IV. or Case III.

CASE XIV.

When the ordinate and semi-curve are given,

Divide the latter by the former, and find the quotient in column $\frac{z}{y}$ of the table, then, opposite, in column φ, will be found the angle at the point of suspension, by which, in conjunction with either of the given quantities, the remaining ones may be found by Case V. or Case III.

CASE XV.

When the absciss and ordinate are given, Divide the latter by the former, and find the quotient in column $\frac{y}{x}$ of the table; then, opposite, in column φ, will be found the angle at the point of suspension, by which, in conjunction with either of the given quantities, the remaining ones may be found by Case IV. or Case V.

The use of the table in drawing the Catenary.

Let it be required to construct a catenary whose absciss = 120, and ordinate 200.

We must first find the tension at the vertex corresponding to the given ordinate and absciss.

By Case XV. $\frac{200}{120} = \frac{20}{12} = 1.667$, which, sought in column $\frac{y}{x}$ of the table, is found to fall between *52°.42′* and *52°.57′* in column φ: the number in column $\frac{y}{x}$ next less to 1.667 being 1.657 and the next greater 1.671.

Then 1.671 − 1.657 : *52°.57 − 52°.42′*

$:: 1.671 - 1.667$ or $.014 : 15' :: .004 : \frac{15 \times .004}{.014} = 4'$.

Hence $52^\circ 42' + 4' = 52^\circ 46'$ = the angle at the point of suspension. Again, the number in column x opposite 1.671 in column $\frac{y}{x}$ is .65, and that opposite 1.657 is .66; consequently $.66 - .65 = .01$ = the difference of the two numbers in column x, corresponding to $1.671 - 1.657 = .014$ in column $\frac{y}{x}$.

Hence $.014 : .01 :: .004 : .0029$ and $.65 + .0029 = .6529$ = the number in column x, corresponding to 1.667 in column $\frac{y}{x}$, or to $52^\circ 46'$ in column ϕ. Again, by Case III, $\frac{120}{.6529} = 183.8$ = the tension at the vertex of the proposed catenary.

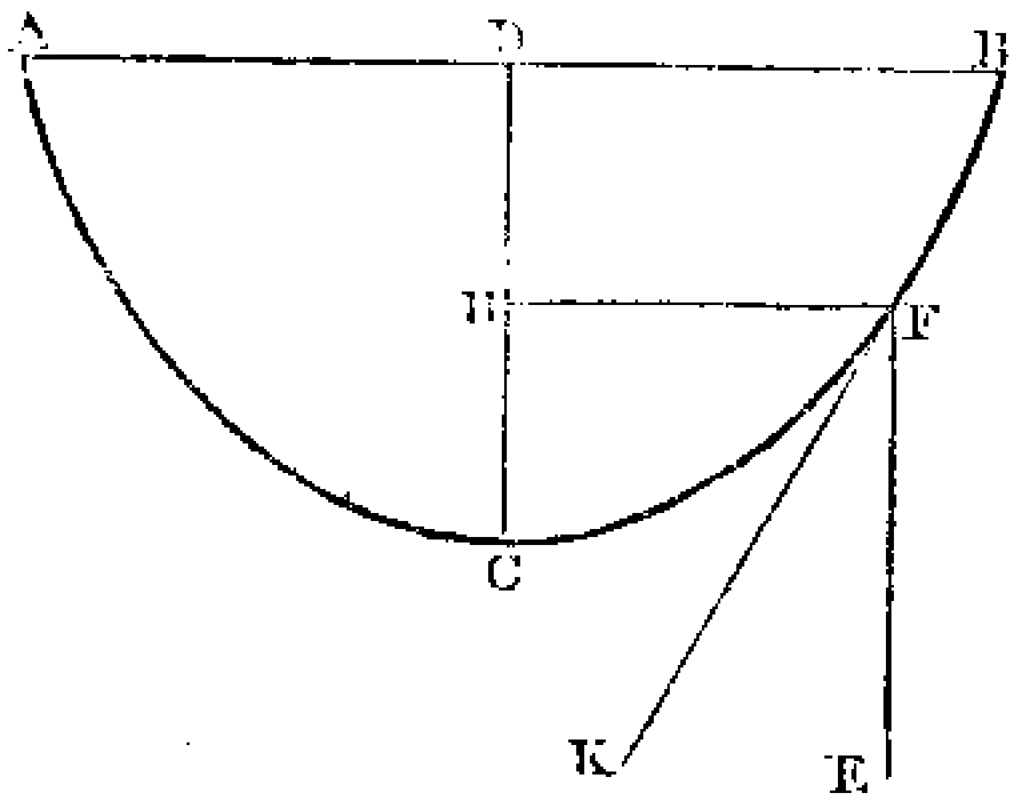
Take now from any scale of equal parts Fig. 23.
AB=120, and, from the same scale, lay off from B the perpendicular BC=200. Then assume ϕ equal to any number of degrees, as 10°, and take out from the table opposite $\phi = 10^\circ$, from column x and y the

numbers .01543 and .17542, and multiply each of them by the tension at the vertex 183.8 and the results will be 2.836 and 32.24. Lay off AD=2.836, and from D parallel to B C lay off DE=32.24, then shall E be a point in the curve. Assume ϕ again equal to any other number of degrees (which is contained in the table) as 18°,40′, and from columns x and y opposite ϕ 18°,40′ take out the numbers .05555 and .33171, which multiply by the tension at the vertex, and the results will be 10.21 and 60.97. Lay off AF=10.21, and through F parallel to BC lay off FG=60.97, then shall G be another point in the curve. In the same manner, by assuming different values of ϕ in succession we may find as many points I, L, &c. in the curve as we please; and hence a curve AEGILC, drawn through all the points so found, will be the catenary required.

OF THE

CATENARY.

1. **Catenary**, or Catenaria, is the name given to the curve formed by a flexible string, rope, or chain, hanging freely, by the power of gravity from any two points of suspension, as ACB.



2. If the string, rope, or chain, be of uniform density and thickness, the curve it forms is called the *simple* or *common* catenary; and is that treated of in this section.

3. The lowest point (C) in the curve is termed the *vertex* or *apex*, and a straight

line (CD) drawn through the vertex perpendicular to the horizon, is termed the *axis.* A horizontal line (HF) drawn from any point (H) in the axis to meet the curve, is called an *ordinate*, and the distance (CH) between the vertex and the ordinate is called *abscissa*, or *absciss.*

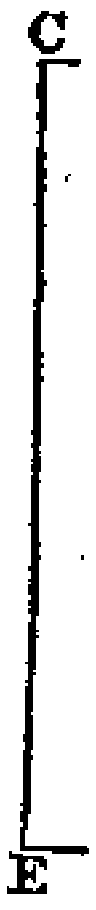
4. Theorem. If any point F in the chain ACB be supported in the position it has taken by the formation of the catenary ACB, the part BF, above that point, may be removed without altering either the position or the tension of any point in the remaining part ACF.

For it is manifest that B F has no other effect upon ACF than by cohesion to support the point F in the position which it has assumed in the formation of the catenary; and if that point be supported in the same position by any other power, it follows that no alteration can take place either in the position or tension of any point in ACF, by the exchanging of similar and equivalent powers.

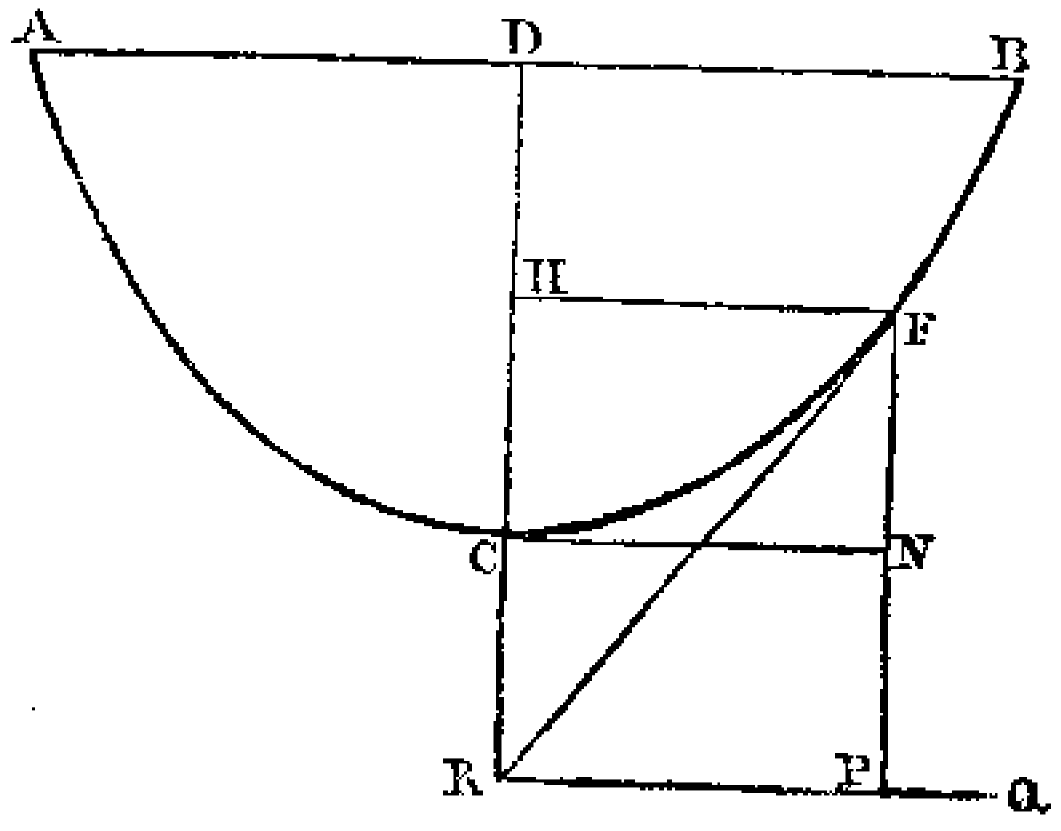
5. Corollary. From this theorem it is manifest that if the position of the support-

ing point F be taken at the vertex C, and then be made to move in the direction which the curve would form if suspended from the point B to any other position, the tension at the point C must remain constant.

6. It may not be superfluous here to remark, that if a chain C E be suspended vertically, the tension at the upper point C is equivalent to the weight of that chain; and that if the chain be of a uniform density and thickness, but of variable length, the tension at the upper point will always be as the length; because such tension being equal to the weight is always equal to the length multiplied by the specific gravity per linear unit.

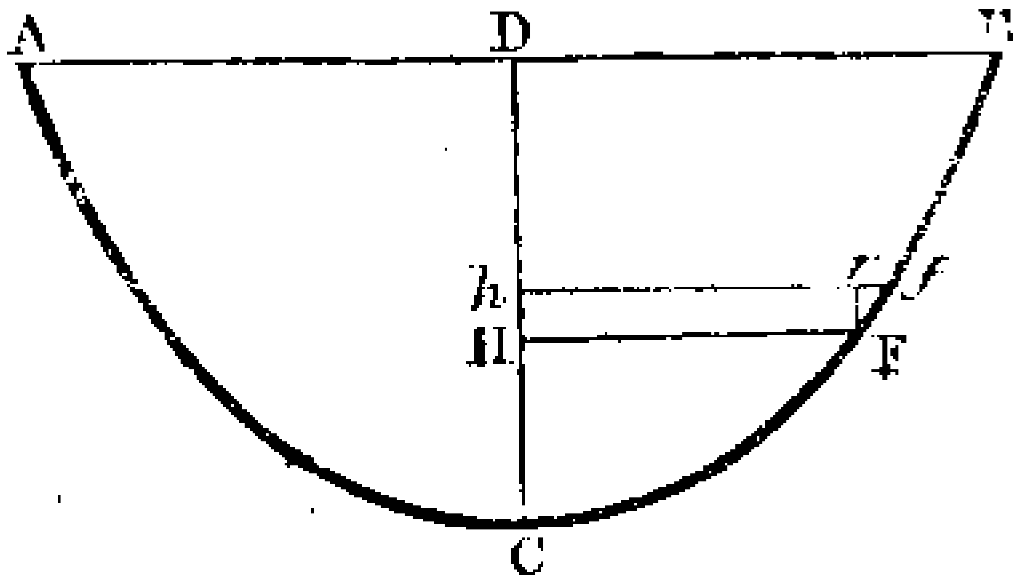


7. Hence the *length* of a chain F P,



similar to that composing the catenary, suspended to any point F in the curve, and producing by its weight the same tension at that point in the direction of gravity, as that of the curve at the same point in the direction of its tangent, is a proper *measure of the tension* of the chain at that point; in like manner the length of a similar chain C R may be taken as a proper measure of the constant tension at the vertex C.

8. Problem. Wherein it is required to investigate the relations between the different parts of the catenary; which relations are commonly called the *equations of the curve.*



Let ACB be a catenary whose vertex is C, and axis CD; and conceive the *variable* ordinate HF to move from the point C, always parallel to itself, and suppose *hf* to be a position of such ordinate indefinitely near to HF; then through F draw Fr parallel to CD; and put the absciss CH $=x$, the ordinate HF$=y$, the arc CF$=z$ the angle CFH$=\phi$, the *measure* of the constant tension at the vertex, as explained by art. 6. and 7.$=c$, and the *measure* of the tension at F$=t$, and the specific gravity of the chain per linear unit$=g$.

Then by art. 6. and 7. will the tension at C$=cg$, that at F$=tg$, and the weight of the part CF of the chain $= zg$: also the three sides rF, Ff and fr of the nascent rectilineal triangle rFf will be respectively represented by $\dot{x}$, $\dot{z}$, and $\dot{y}$. Now, the chain having assumed the form ACB, it is immaterial whether the part CF be considered flexible or rigid. On the latter supposition it is kept at rest by the three following forces; viz. the tension at C acting in the horizontal direction of rf; that at F acting in the direction of the tangent at the point F, or of the indefinitely small arc Ff; and the weight of CF, acting in the vertical direction of rF. And since we have shown that these forces are represented in magnitude by cg, tg, and zg, respectively, and that the directions in which they act, are respectively parallel to the three sides $\dot{y}$, $\dot{z}$, and $\dot{x}$, of the triangle frF, it follows, by the laws of statics, that cg, tg, and zg, and consequently c, t, and z, must be to each other as $\dot{y}$, $\dot{z}$, and $\dot{x}$: whence we have $\frac{\dot{y}}{c}$, $\frac{\dot{z}}{t}$ and $\frac{\dot{x}}{z}$ all equal to each other.

Again, the indefinitely small arc Ff, being considered as a straight line, intersecting the two parallel lines hf, HF, we have (by Euc. i. 29.) the angle rfF equal to the exterior angle CFH $= \varphi$, and (by Euc. i. 47.) $\mathrm{F}f^2 = \mathrm{F}r^2 + rf^2$ or $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$: and since rF is the tangent, and fF the secant of the angle rfF to the radius rf, we have, by putting radius $= 1$, $\dot{y}$, $\dot{x}$, and $\dot{z}$, to each other respectively, as 1, tang.φ, and sec.φ. And since it has been shown, that $\dot{y}$, $\dot{x}$, and $\dot{z}$ are to each other as c, z, and t; it follows that c, z, and t must be to each other as 1, tang. φ, and sec.φ, or that c, $\frac{z}{\text{tang.}\varphi}$ and $\frac{t}{\text{sec.}\varphi}$ must all be equal to each other.

Hence $z = c$ tang.ϕ, and $t = c$ sec. ϕ. Moreover by comparing the equal quantities $\frac{\dot{y}}{c}$ and $\frac{\dot{x}}{z}$ we have $\dot{y} = \frac{c\dot{x}}{z}$ or by squaring each side, and adding $\dot{x}^2$ to each side of the result, we obtain $\dot{x}^2 + \dot{y}^2 = \dot{x}^2 + \frac{c\,\dot{x}^2}{z^2} = \frac{\dot{x}^2}{z^2}(z^2 + c^2)$; or since $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$; $\dot{z}^2 = \frac{\dot{x}^2}{z^2}(z^2 + c^2)$ or $\dot{z} = \frac{\dot{x}}{z}\sqrt{z^2 + c^2}$; hence by substituting this value of $\dot{z}$ in the equation $\frac{\dot{z}}{t} = \frac{\dot{x}}{z}$ we obtain $\frac{\dot{x}\sqrt{z^2 + c^2}}{t\,z} = \frac{\dot{x}}{z}$; or by dividing by $\frac{\dot{x}}{t\,z}$, we have $\sqrt{z^2 + c^2} = t$, and consequently $c = \sqrt{t^2 - z^2}$, and $z = \sqrt{t^2 - c^2}$.

Again from the equation $\frac{\dot{z}}{t} = \frac{\dot{x}}{z}$ we have $\dot{x} = \frac{z\dot{z}}{t}$, which, by substituting $\sqrt{z^2 + c^2}$ for t, becomes $\dot{x} = \frac{z\dot{z}}{\sqrt{z^2 + c^2}}$, or by taking the *correct* fluent of each side, (observing that x and z are both $= o$ at the same time) we obtain $x = \sqrt{z^2 + c^2} - c$. Hence $x + c = \sqrt{z^2 + c^2}$, from which $z = \sqrt{2cx + x^2}$, and by substituting t for $\sqrt{z^2 + c^2}$ in the equation $x + c = \sqrt{z^2 + c^2}$, we have $x + c = t$ or $x = t - c = c$ sec.$\phi - c = c($sec.$\phi - 1) = \frac{c \text{ ver. sin.}\phi}{\text{cos. }\phi}$

Lastly from the equation $\frac{\dot{y}}{c} = \frac{\dot{z}}{t}$ we have $\dot{y} = \frac{c\dot{z}}{t}$, which, by substituting $\sqrt{z^2+c^2}$ for t, becomes $\dot{y} = \frac{c\dot{z}}{\sqrt{z^2+c^2}}$; or, by taking the correct fluent of each side (observing that y and z begin together) $y = c.\text{hyp.log.}\frac{z+\sqrt{z^2+c^2}}{c}$, which, by subst. $= c.\text{hyp.log.}\frac{\sqrt{2cx+x^2}+c+x}{c}$ $= c.\text{hyp.log.}\frac{z+t}{c} = c.\text{hyp.log.}(\text{tang.}\,\varphi + \text{sec.}\,\varphi)$ $= c.\text{hyp.log.tang.}\ (45^\circ + \frac{1}{2}\varphi)$.

Or since, by nature of logarithms, the hyperbolic logarithm of any number is equal to 2.302585 multiplied by the common tabular logarithm of that number; by putting $2.302585 = m$ and the word log. to express the common tabular logarithm, the foregoing equations may be transformed into the following, viz. $y = cm\ \text{log.}\frac{z+\sqrt{z^2+c^2}}{c}$ $= cm\ \text{log.}\ \frac{\sqrt{2cx+x^2}+c+x}{c} = cm\ \text{log.}\ \frac{z+t}{c} =$ $cm\ \text{log.}\ (\text{tang.}\ \varphi + \text{sec.}\ \varphi) = cm\ \text{log. tang.}$ $(45^\circ + \frac{1}{2}\varphi)$. But it is to be observed, that in taking the logarithms from a *table of logarithmic sines, tangents, secants*, &c., 10 *must* *be deducted from the index*, because in the following investigation the radius is made 1, the logarithm of which is 0, whereas in the table of logarithmic sines, tangents, &c. the logarithm of the radius is made 10.

By the different relations obtained in this article we are furnished with sufficient data to solve the following:

9. Problem, wherein it is proposed to find any four of the six quantities φ, c, t, z, x, and y, the remaining two being given.

CASE I.

When φ and c are given, to find the rest.

$$z = c \text{ tang. } \varphi$$

$$t = c \text{ sec. } \varphi$$

$$x = c\ (\text{sec.}\varphi - 1) = \frac{c \text{ ver.sin.}\phi}{\text{cos. } \phi}$$

$$y = c\, m \text{ log.tang. } (45^\circ + \tfrac{1}{2}\varphi)$$

CASE II.

When φ and t are given, to find the rest.

$$c = \frac{t}{\text{sec. } \phi} = t \text{ cos. } \varphi$$

$$z = t \text{ sin. } \varphi$$

$$x = t \text{ ver. sin. } \varphi$$

$$y = mt \text{cos.}\varphi \text{ log.tang.} (45^\circ + \tfrac{1}{2}\varphi)$$

CASE. III.

When ϕ and z are given, to find the rest.

$$c = z \text{ cotang. } \phi$$

$$t = z \text{ cosec. } \phi$$

$$x = z \text{ cosec. } \phi \text{ ver. sin. } \phi = \frac{z \text{ ver.sin. } \phi}{\text{sin. } \phi}$$

$$y = m\, z \text{ cot. } \phi \text{ log. tang. } (45^\circ + \tfrac{1}{2}\phi)$$

CASE. IV.

When ϕ and x are given, to find the rest.

$$c = \frac{x}{\text{sec. } \phi - 1} = \frac{x \text{ cos. } \phi}{\text{ver. sin. } \phi}$$

$$t = \frac{x}{\text{ver. sin. } \phi.}$$

$$z = \frac{x \text{ sin. } \phi}{\text{ver. sin. } \phi}$$

$$y = \frac{m\, x \text{ cos. } \phi \text{ log. tang. } (45^\circ + \tfrac{1}{2}\phi)}{\text{ver. sin. } \phi.}$$

CASE V.

When ϕ and y are given, to find the rest.

$$c = \frac{y}{m \text{ log. tang. } (45^\circ + \tfrac{1}{2}\phi)}$$

$$t = \frac{y}{m \text{ cos. } \phi \text{ log. tang. } (45^\circ + \tfrac{1}{2}\phi)}$$

$$z = \frac{y \text{ tang. } \phi}{m \text{ log. tang. } (45^\circ + \tfrac{1}{2}\phi)}$$

$$x = \frac{y \text{ ver sin. } \phi}{m \text{ cos. } \phi \text{ log. tang. } (45^\circ + \tfrac{1}{2}\phi)}$$

CASE VI.

When c and t are given, to find the rest.

$x = t - c.$

$z = \sqrt{t^2 - c^2}$

Sec. $\varphi = \frac{t}{c}$

$y = c\ m \log. \frac{t + \sqrt{t^2 - c^2}}{c}$

CASE VII.

When c and z are given, to find the rest.

$t = \sqrt{z^2 + c^2}$

$x = \sqrt{z^2 + c^2} - c$

Tan. $\varphi = \frac{z}{c}$

$y = c\ m \log. \frac{\sqrt{z^2 + c^2} + z}{c}$

CASE VIII.

When c and x are given, to find the rest.

$t = c + x.$

$z = \sqrt{2cx + x^2}$

See $\varphi = 1 + \frac{x}{c}$

$y = cm \log. \frac{c + x + \sqrt{2cx + x^2}}{c}$

CASE IX.

When c and y are given, to find the rest. Log. tang. $(45^\circ + \frac{1}{2}\varphi) = \frac{y}{c\,m}$ from which φ may be found; and when φ is found, t, z, and x may be found as in Case I.

CASE X.

When t and z are given, to find the rest.

$$c = \sqrt{t^2 - z^2}$$

$$x = t - \sqrt{t^2 - z^2}$$

$$\text{Sin. } \varphi = \frac{z}{t}$$

$$y = m\sqrt{t^2 - z^2}\ \text{log.}\ \frac{t+z}{\sqrt{t^2 - z^2}}$$

CASE XI.

When t and x are given, to find the rest.

$$c = t - x.$$

$$z = \sqrt{2tx - x^2}$$

$$\text{Sec. } \varphi = \frac{t}{t-x}$$

$$y = m\,(t - x)\ \text{log.} \frac{t + \sqrt{2tx - x^2}}{t - x}$$

CASE XII.

When t and y are given, to find the rest.

Cos. φ log. tang. $(45^\circ + \frac{1}{2}\varphi) = \frac{y}{m\,t}$, from which φ may be found by approximation; and φ being found, c, z, and x may be found as in Case II.

Note. In this case we must observe that φ may have two different values, the one below and the other above 56°28, as will appear from what follows; and as φ may have two values, so c, z, and x must have each of them two corresponding values.

CASE XIII.

When z and x are given, to find the rest.

$$c = \frac{z^2 - x^2}{2\,x}$$

$$t = \frac{z^2 + x^2}{2\,x}$$

$$\text{Sec.}\varphi = \frac{z^2 + x^2}{z^2 - x^2}$$

$$y = m\left(\frac{z^2 - x^2}{2\,x}\right)\log.\frac{z + x}{z - x}$$

CASE XIV.

When z and y are given, to find the rest.

Cot. φ log. tang. $(45^\circ + \frac{1}{2}\varphi) = \frac{y}{m\,z}$ from which φ may be found by approximation;

and φ being found, c, t, and x may be determined byCase III.

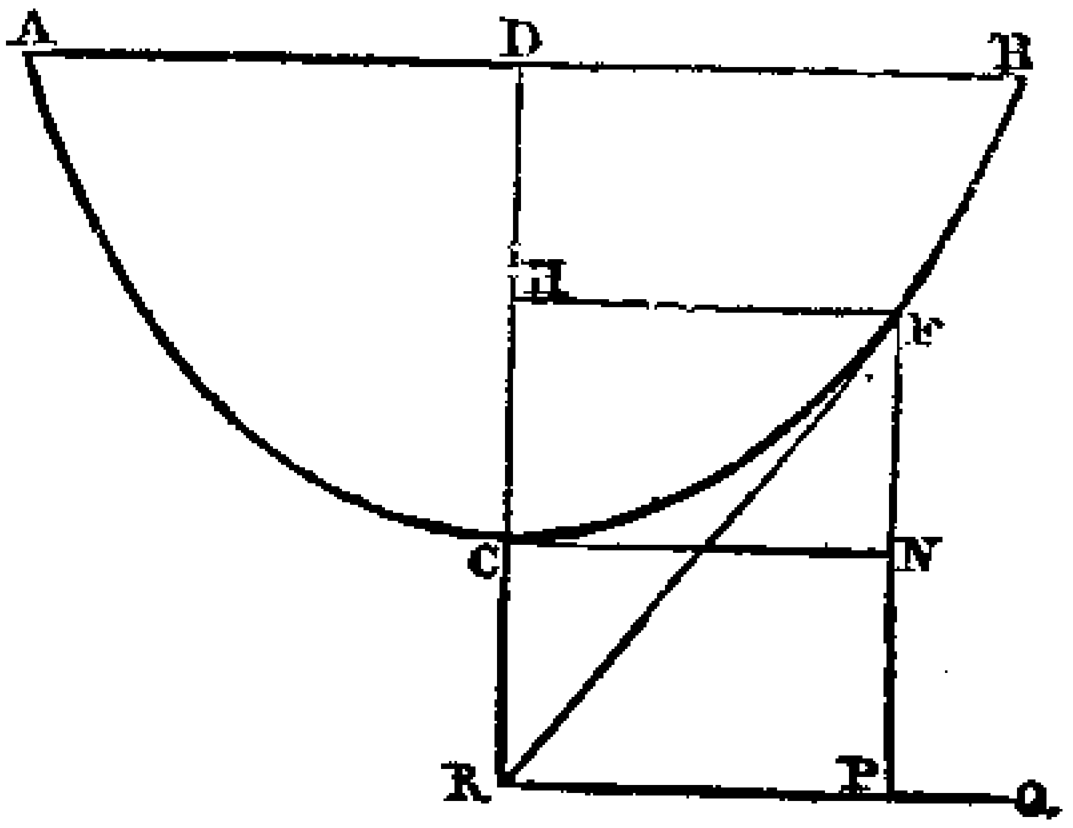
CASE XV.

When x and y are given, to find the rest. $\frac{\text{Log.tang,}(45^\circ+\frac{1}{2}\phi)}{\text{sec.}\phi\,\text{ver.sin.}\phi} = \frac{y}{mx}$ from which φ may be found by approximation; and when φ is found, c, t, and z may be determined, as in Case III.

10. Since it has been shown in Art. 8. that $\dot{y} = \frac{c\dot{x}}{z}$ and, in receding from the vertex, $\frac{c\dot{x}}{z}$ is always positive, $\dot{y}$ must also be a positive quantity; consequently y must always increase, and the curve must therefore continually recede from its axis.

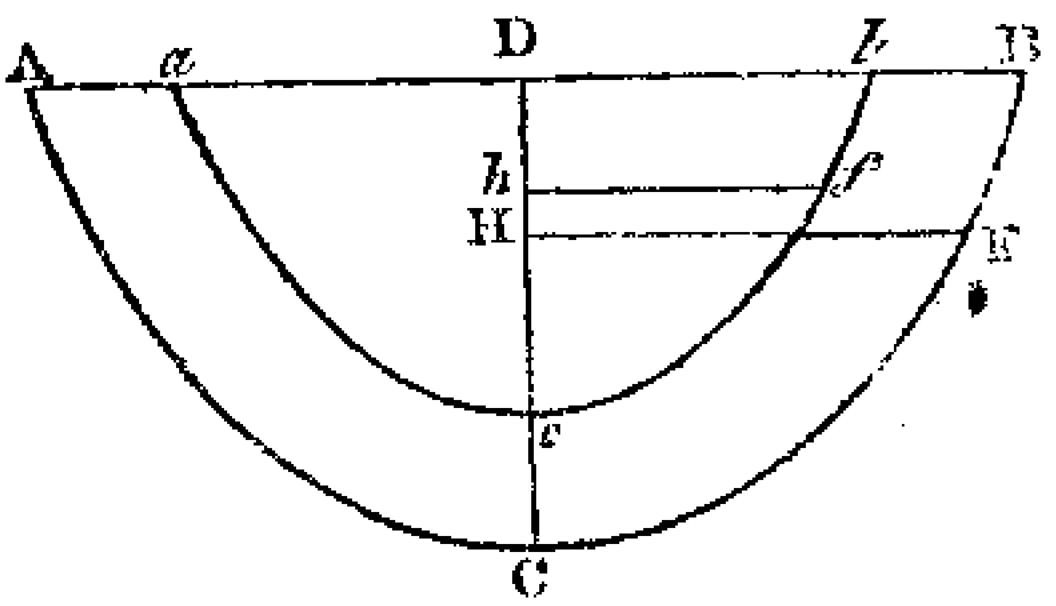
11. From the same equation it evidently appears that $\dot{y} \propto \frac{c\dot{x}}{z} \propto \frac{\dot{x}}{z}$ or $\frac{\dot{y}}{x} \propto \frac{1}{z}$, and since z is always a variable quantity, $\frac{\dot{y}}{\dot{x}}$ must also be variable, and consequently the catenary has no point of contrary flexture; and since the curve is always concave towards its axis, it follows also that, in receding from the vertex, the ratio of y to x continually diminishes.

12. From the equation $t=c+x$ obtained in article 8., we have the following method of expressing geometrically the tension at any point in the curve of a catenary drawn from a given tension at the vertex. From the vertex C, draw a vertical line CR equal to the *measure* of the given tension at the vertex, as explained in article 7., and through its lower extremity R, draw the horizontal line RQ; then if from any point F in the curve a line FP be drawn parallel to CR, and terminated in RQ, FP will express the *measure* of the tension at the point F. For produce RC to form the axis CD, and and draw the ordinate FH; then, by article 8., CR $= c$, CH$=x$, and HR F P $=c+x=t$.



Hence the horizontal line RQ may be called a *directrix* to the catenary.

13. *Theorem.* All catenary curves making equal angles with their ordinates at the points of suspension are similar.



Let ACB, *acb*, be any two catenary curves having DB and D*b* for their ordinates at the points of suspension B and *b*, and having the angles DBC, D*bc* equal, then will ACB and *acb* be similar.

For let the angle DBC $=$ D*bc* $= \varphi$, the tension at C $= c$, that at B$=t$, CB $= z$, CD $= x$ and DB $= y$, and the corresponding quantities in the catenary D*bc*, $= c'$, t', z', x', and y'.

Then by Case I. article 9.

$$z = c \text{ tang. } \varphi, \quad z' = c' \text{ tang.} \varphi \therefore \frac{z}{c} = \frac{z'}{c'}$$

each $=$ tang.φ

$$t = c \text{ sec. } \varphi, \quad t' = c' \text{ sec. } \varphi \therefore \frac{t}{c} = \frac{t'}{c'}$$

each $=$ sec. φ

$$x = c(\text{sec.}\phi - 1), \quad x' = c'(\text{sec.}\phi - 1)$$

$\therefore \frac{x}{c} = \frac{x'}{c'}$ each $=$ sec. $\phi - 1$.

$y = cm$ log. tang. $(45^\circ + \frac{1}{2})$ and $y' = cm$.log.tang.$(45^\circ + \frac{1}{2}\phi)$ $\therefore \frac{y}{c} = \frac{y'}{c'}$ each $= m$ log. tang.$(45^\circ + \frac{1}{2}\phi)$

Hence $c : c' :: \left\{ \begin{array}{c} z : z' \\ t : t' \\ x : x' \\ y : y' \end{array} \right\}$ from which the proposition is manifest.

14. From art. 9. Case V. we have.

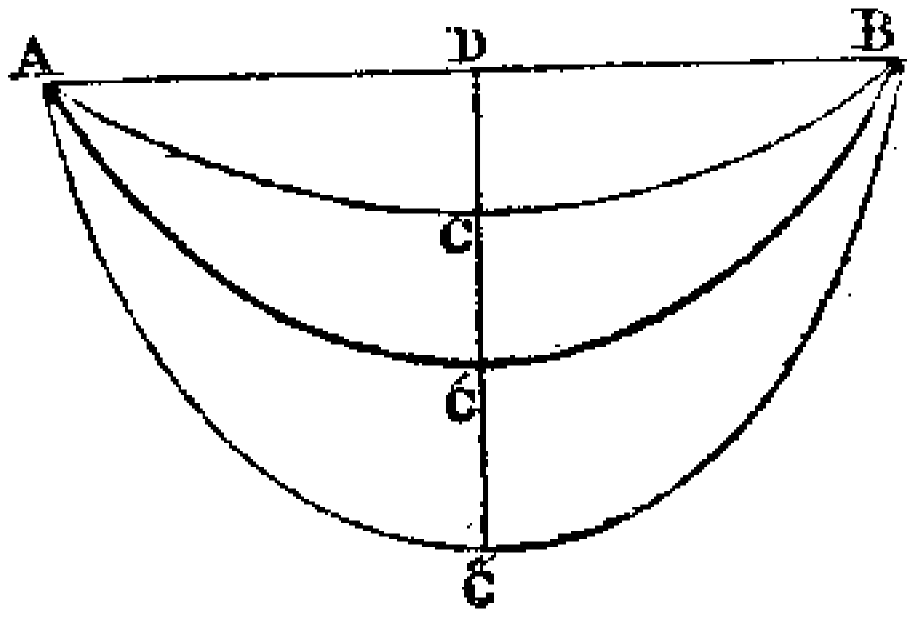
$$t = \frac{y}{\text{m.cos.}\phi\text{log.tang.}(45^\circ + \frac{1}{2}\phi)} = \frac{y\text{sec.}\phi}{m\text{log.tang.}(45^\circ + \frac{1}{2}\phi)}$$

and if in this last expression we make y constant and $\phi = 0$, we have

$t = \frac{y \times 1}{m\text{log.}1} = \frac{y}{m \times 0} = \frac{y}{0} =$ infinity, and, still making y constant, if ϕ be taken indefinitely near to 90°, $45^\circ + \frac{1}{2}\phi$ will also be indefinitely near to 90°: and, in that case, sec.ϕ and tang.$(45^\circ + \frac{1}{2}\phi)$ must, by Trigonometry, both be indefinitely great, and be to each other in a ratio indefinitely near to that of equality. Hence it is manifest that t may then be taken $= \frac{y\text{sec.}\phi}{m\text{log.sec.}\phi}$

But, by the nature of logarithms, a number indefinitely great is infinitely greater than its logarithms; consequently t must, in that case also, be indefinitely great. Since, therefore, t is indefinitely great when ϕ is taken either $= 0$, or indefinitely near to 90°; and that between these limits t is of a finite magnitude, it follows that, by making y constant, some value of ϕ may be found at which t is a *minimum*. From this article naturally arises the following:

15. Problem. Given two points A, B, in the same horizontal line, and a flexible chain of an unlimited length fixed at one of them, but moving through the other so as to form successively the different figures ACB, AC′B, AC″B, &c., to find the angle ABC when the tension at B is a *minimum*.



Since $t = \frac{y}{m\cos.\phi\log.\text{tang}.(45^\circ+\frac{1}{2}\phi)}$, t a *minimum*, and y constant, we have $\frac{1}{m\cos.\phi\log.\text{tang}.(45^\circ+\frac{1}{2}\phi)} =$ a *minimum*.

Hence, its reciprocal, $m\cos.\phi\log.\text{tang}.(45^\circ+\frac{1}{2}\phi) = \cos.\phi\,\text{hyp.log.tang}.(45^\circ+\frac{1}{4}\phi)$ must be a *maximum*.

Whence $\dot{\text{cos}}.\phi\,\text{hyp.log.tang}.(45^\circ+\frac{1}{2}\phi) + \cos.\phi,\dot{F}.\text{hyp.log.tang}.(45^\circ+\frac{1}{2}\phi) = 0$.

But $\cos.\phi\,\dot{F}.\text{hyp.log.tan}.(45^\circ+\frac{1}{2}\phi) = \frac{\cos.\phi}{\sin.\phi}$; consequently, $\dot{\text{cos}}.\phi\,\text{hyp.log.tang}.(45^\circ+\frac{1}{2}\phi) - \frac{\dot{\text{cos}}.\phi}{\sin.\phi} = 0$

Or $\text{hyp.log.tang}.(45^\circ+\frac{1}{2}\phi) = \frac{1}{\sin.\phi} = \text{cosec}.\phi$

Or $m\log.\text{tang}.(45^\circ+\frac{1}{2}\phi) = \text{cosec}.\phi$

Whence, $\frac{\text{cosec}.\phi}{\log.\text{tang}.(45^\circ+\frac{1}{2}\phi)} = m$. from which ϕ may be found by approximation.

In order to determine ϕ from this expression, we shall use Mr. Bonnycastle's rule of Double Position, after first finding, by trial, that the value of ϕ lies between 50° and 60°.

Suppose $\varphi = 50^\circ$, then

$$\frac{\text{cosec.}\ \phi}{\text{log.tang.}(45^\circ+\frac{1}{2}\varphi)} = \frac{\text{cosec.}\ 50^\circ}{\text{log.tang.}70^\circ} = \frac{1.305407}{.438934}$$

$= 2.97407.$

And suppose $\varphi = 60^\circ$, then

$$\frac{\text{cosec.}\ \phi}{\text{log.tang.}(45^\circ+\frac{1}{2}\phi)} = \frac{\text{cosec.}\ 60^\circ}{\text{log.tang.}75^\circ} = \frac{1.1547}{.571947}$$

$= 2.01889.$

But the true value of $\frac{\text{cosec.}\ \phi}{\text{log.tang.}(45^\circ+\frac{1}{2}\phi)} = m$
$= 2.302585.$

Whence $2.97407 - 2.01889 : 60^\circ - 50^\circ$
$:: \ 2.97407 - 2.302585$, or $.95518 : 10^\circ$
$:: \ .671485 : 7^\circ$ nearly.

Consequently $\varphi = 50^\circ + 7^\circ = 57^\circ$ nearly.

Again, suppose $\varphi = 57^\circ$ then

$$\frac{\text{cosec.}\ 57^\circ}{\text{log.tang.}73^\circ.30'} = \frac{1.192363}{.528395} = 2.256576$$

And suppose $\varphi = 56^\circ$, then

$$\frac{\text{cosec.}\ 56^\circ}{\text{log.tang.}73^\circ} = \frac{1.206218}{.514661} = 2.343714$$

Then $2.343714 - 2.256576 : 57^\circ - 56^\circ$
$:: \ 2.343714 - 2.302585$, or $.087138 : 1^\circ$
$:: \ .041129 : \frac{.041129^\circ}{.087138} = \frac{.041129 \times 60'}{.087138}$
$= \frac{2.46774'}{.087138} = 28'$. Wherefore $\varphi = 56^\circ + 28'$

$= 56° 28' =$ the angle ABC which was to be found.

16. Corollary. If in the expression m log. tang. $(45°+\frac{1}{2}\varphi) = \text{cosec.}\ \varphi = \frac{1}{\text{sin.}\,\phi}$ obtained in the last problem, we substitute $\frac{y}{cm}$ for log. tang. $(45°+\frac{1}{4}\varphi)$ according to Case IX. of art. 9. and for $\frac{1}{\text{sin.}\ \phi}$ we substitute $1 \div \frac{z}{t} = \frac{t}{z}$ according to Case X. of the same article, we have $\frac{y}{c} = \frac{t}{z}$ or $y = \frac{ct}{z}$ for a particular relation between y, c, t, and z, which obtains only when t is a *minimum* with respect to y.

17. Problem. Wherein it is proposed to find the subtangent to the catenary.

From article 8, we have $\dot{y} = \frac{c\dot{x}}{z}$ or $\frac{\dot{x}}{\dot{y}} = \frac{z}{c}$ whence, by multiplying each side by y, $\frac{\dot{x}y}{\dot{y}} = \frac{zy}{c} =$ the subtangent required.

18. Corollary. By putting s for the subtangent, we have $s = \frac{zy}{c}$; and from article 16. when t is a *minimum* with respect

to y, $y = \frac{ct}{z}$. Hence by multiplying these equations together, we obtain $sy = \frac{ctzy}{cz} = ty$ or $s = t$.

Whence it appears that when t is a *minimum* with respect to y, the tangent FR cuts the axis DC produced, at the same point as the directrix RQ does. Fig. Page 115

19. Problem. Given any one of the five quantities c, x, t, z, y, to find the other four geometrically, when t is a *minimum* with respect to y.

CASE I.

When c is given, to find the rest.

Fig. 24. From any point A in the straight line AC, draw AD, making the angle CAD= 56° 28′, on AC take AB= the given quantity c, and through B draw BD perpendicular to AC cutting AD in D: with the centre A and distance AD describe an arc FDC, cutting AC in C, and through C draw CE parallel to BD.

Lastly, through A draw AE perpendicular to AD, meeting CE in E; then shall CB=x, BD=z, AD or AC=t and CE=y.

For by art. 15., the angle φ when t is a *minimum* with respect to $y = 56°.28'$.

But by construction the angle DAB $= 56°.28'$; consequently DAB $= \varphi$.

Hence, in the triangle DAB we have : rad. : AB : : sec. DAB : AD : : tang. DAB : BD, or, radius being unity, $1 : c : :$ sec. φ : AD $= c$ sec. φ; and $1 : c : :$ tang. φ : BD $= c$ tang. φ.

But by Case I. art. 9., $t = c$ sec. φ, and $z = c$ tang. φ. Hence it follows that AD $= t$ and BD $= z$.

Again, because AC $=$ AD $= t$ and AB $= c$, BC $= t - c = x$ by Case VI. art. 9. Moreover, because in the triangles ADB ACE the angles ABD, ACE are equal, each being a right angle, and that DAB and AEC are also equal, each being the complement of the angle CAE; these triangles are equiangular, and consequently DB : AB : : AC : CE or $z : c : : t$: CE $= \frac{tc}{z}$ which by art. 16. $= y$.

Another method of construction. — On any point in the vertical line KC erect the perpendicular KH, which make equal to c; and through H draw HA, making the angle $H = 56° 28'$, and meeting KC in A; make $AB = KH$ and $AC = AH$, then produce HA to meet CE, drawn parallel to KH in E, then shall B be the vertex, and E another point in the catenary curve BE. For by comparing the triangle HAK with with ABD, drawn by the other method, we have $HK = AB$, the angle $H = DAB$, $\text{each} = 56\ 28'$, the angle $K = ABD$, and consequently the triangles are equal in all respects; hence $AH = AD = t$, and $AK = BD = z$, the whole KC is therefore $= t + x$.

CASE II.

When x is given, to find the rest.

Make $BC = x$, and through the point C draw CD, making the angle $BCD = \frac{180 - 56° 28'}{2} = \frac{123° 32'}{2} = 61° 46'$, meeting the perpendicular BD in D; through D draw DA, making the angle CDA equal to the angle BCD, and meeting CB produced in A. Then through A draw AE perpendicular to AD, meeting CE, drawn parallel to

DB, in E, and the construction is done; for then BD, AD, AB, and CE respectively represent z, t, c, and y, as in Case I.

CASE III.

When t is given, to find the rest.

From the point A in the straight line AC taken $= t$, draw AD $= t$, making the angle CAD $= 56^\circ\ 28'$; through D draw DB perpendicular to AC in B, and through A draw AE perpendicular to AD meeting CE, drawn through C parallel to DB, in E.

CASE IV.

When z is given, to find the rest.

From any point B in the indefinite straight line AC, draw the perpendicular BD $= z$, and through D draw DA, making the angle BDA $= 90^\circ - 56^\circ\ 28' = 33^\circ\ 32'$, and meeting AC in A. With the centre A and distance AD describe the arc DC, cutting AC in C, and through A draw AE perpendicular to AD, meeting CE as before.

CASE V.

When y is given, to find the rest.

From the point C in the straight line AC draw the perpendicular CE$=y$, and through E draw EA, making the angle CEA $=$ 56° 28′, meeting AC in A. With the centre A and distance AC describe the arc CDF, and through A draw AD perpendicular to AE, meeting the arc CDF in D; then from D let fall DB perpendicular to AC, and the thing is done.

This problem, which is of a limited application, leads us to the contemplation of another somewhat more general.

20. Problem. Wherein it is proposed to find geometrically any three of the five quantities φ, c, x, t, and z, the remaining two being given.

CASE I.

When φ and c are given, make AB$=c$, and the angle BAD$=\varphi$, and complete the figure ADCB as in Case I. art. 19., then shall BC$=x$, AD or AC$=t$, and BD$=z$.

CASE II.

When φ and x are given, make BC$=x$, and the angle BCD$=\frac{1}{2}(180^\circ-\varphi)$ and complete the figure ADC as before.

CASE III.

When φ and t are given, make AC$=t$, and the angle CAD$=\varphi$, and proceed as before directed.

CASE IV.

When φ and z are given, make BD$=z$, and the angle BDA$=90^\circ-\varphi$, and complete the figure as before.

CASE V.

When c and x are given, make AB$=c$, BC$=x$, draw the perpendicular DB, and with the centre A and distance AC, describe the arc CDF cutting BD in D, join AD, then will AC or AD$=t$, BD$=z$, and the angle CAD$=\varphi$.

CASE VI.

When c and t are given, make AC$=t$, and AB$=c$, and then proceed as in the last case.

CASE VII.

When c and z are given, make AB$=c$, and from B perpendicular to AB draw BD $=z$, then proceed as in Case I.

CASE VIII.

When x and t are given, make AC$=t$, CB$=x$, and then proceed as in Case V.

CASE IX.

When x and z are given, make CB$=x$, and from B, perpendicular to CB, draw BD $=z$, and join CD; then from D draw DA making the angle CDA=the angle BCD, and meeting CB produced in A, and the thing is done.

CASE X.

When t and z are given, from any point B, in the indefinite straight line AC, draw BD$=z$, and with the centre D and distance DA$=t$ describe an arc cutting AC in A, and still with the same radius, with the centre A cut AC in C, and the construction is finished.

This problem, being evident from what was said in the last, requires no demonstration.

21. Theorem. Let ACB be a catenary whose vertex is C, axis CD, and directrix EH; take any two equal parts FG, GH, on the directrix, and parallel to CD, draw FM, GN, and HB, through F, G, and H, cutting the curve in M, N, and B; through these points again draw the double ordinates MO*m*, NP*n*, and BQA. On the prolongation of FM, GN, and HB, take MI=CM, NK=CN, and BL=CB; join IO, KP, LQ; and produce them to meet the perpendiculars *mf*, *ng*, A*h*, drawn from *m*, *n*, and A, in *i*, *k*, and *l*. Then will CE be a mean proportional between FI and *fi*, between GK and *gk*, and between HL and *hl*; *gk* will be a mean proportional between *fi* and *hl*. Fig. 25.

For since the two triangles IMO, *imo*, are equi-angular, and OM=*om*, MI must be equal to *mi*. In the same manner it may be shown that NK=*nk*, and BL=A*l*. Hence by adopting the same notation as before FM=*fm*=t, and MI=*mi*=CM=z; whence FI=$t+z$ and *fi*=$t-z$, consequently FI×*fi*=$(t+z)\ (t-z)=t^2-z^2$. But by art. 9. Case X., $c^2=t^2-z^2$, consequently FI×*fi*=c^2

$=\mathrm{CE}^2$; or CE is a mean proportional between FI and *fi*. In the same manner it may be shown that CE is a mean proportional between GK and *gk*; and between HL and *hl*.

Again put $\mathrm{OM}=\mathrm{EF}=y$, and FG or GH $=ny$; then $\mathrm{PN}=\mathrm{EG}=\mathrm{EF}+\mathrm{FG}=y+ny=\overline{1+n})y$, and $\mathrm{QB}=\mathrm{EH}=\mathrm{EF}+2\mathrm{FG}=y+2ny=\overline{1+2n})y$. Whence by art. 8. $\mathrm{OM}=y=cm$ $\log.\frac{t+z}{c}=cm\log.\frac{\mathrm{FI}}{\mathrm{CE}}$, and by multiplying each side of this equation by $1+n$, we have $\overline{1+n})y$ $=cm\log.\frac{\mathrm{FI}}{\mathrm{CE}}\times(1+n)$, which by the nature of logarithms $=cm\log.(\frac{\mathrm{FI}}{\mathrm{CE}})^{1+n}$. But $\overline{1+n})y=$ $\mathrm{PN}=cm\log.\frac{\mathrm{GK}}{\mathrm{CE}}$, consequently $cm\log.\frac{\mathrm{GK}}{\mathrm{CE}}=$ $cm\log.(\frac{\mathrm{FI}}{\mathrm{CE}})^{1+n}$; or by dividing each side by cm, we have $\log.\frac{\mathrm{GK}}{\mathrm{CE}}=\log.(\frac{\mathrm{FI}}{\mathrm{CE}})^{1+n}$, or since numbers which have equal logarithms, in the same system, are themselves equal, $\frac{\mathrm{GK}}{\mathrm{CE}}=(\frac{\mathrm{FI}}{\mathrm{CE}})^{(1+n)}$ or $\mathrm{GK}=\mathrm{CE}\,(\frac{\mathrm{FI}}{\mathrm{CE}})^{1+n}$. In the same manner it may be shown that $\mathrm{HL}=\mathrm{CE}(\frac{\mathrm{FI}}{\mathrm{CE}})^{1+2n}$

Hence since $\mathrm{GK}=\mathrm{CE}\ (\frac{\mathrm{FI}}{\mathrm{CE}})^{1+n}=\frac{\mathrm{FI}^{1+n}}{\mathrm{CE}^{n}}$ $\mathrm{GK}^2=\frac{(\mathrm{FI})^{2+2n}}{\mathrm{CE}^{2n}}$. Also $\mathrm{HL}=\mathrm{CE}(\frac{\mathrm{FI}}{\mathrm{CE}})^{1+2n}=\frac{(\mathrm{FI})^{1+2n}}{\mathrm{CE}^{2n}}$; consequently $\frac{\mathrm{GK}}{\mathrm{HL}}=\frac{(\mathrm{FI})^{2+2n}}{\mathrm{CE}^{2n}}\div\frac{(\mathrm{FI})^{1+2n}}{\mathrm{CE}^{2n}}=\frac{\mathrm{FI}^{2+2n}}{\mathrm{FI}^{1+2n}}=\mathrm{FI}$ or $\mathrm{GK}^2=\mathrm{FI}\times\mathrm{HL}$; that is, GK is a mean proportional between FI and HL.

Moreover, from the first part of the demonstration we have $fi=\frac{\mathrm{CE}^2}{\mathrm{FI}}$, $gk=\frac{\mathrm{CE}^2}{\mathrm{GK}}$ and $hl=\frac{\mathrm{CE}^2}{\mathrm{HL}}$: whence fi, $g\,k$, and $h\,l$ are to each other as $\frac{\mathrm{CE}^2}{\mathrm{FI}}$, $\frac{\mathrm{CE}^2}{\mathrm{GK}}$, and $\frac{\mathrm{CE}^2}{\mathrm{HL}}$, or, by dividing each term by CE^2, as $\frac{1}{\mathrm{FI}}$, $\frac{1}{\mathrm{GK}}$, and $\frac{1}{\mathrm{HL}}$.

But it has been shown that FI, GK, and HL are to each other in geometrical progression; consequently fi, gk, and hl must also be to each other in geometrical progression; and gk is therefore a mean proportional between fi and hl. Q.E.D.

22. Corollary. From the above demonstration it is manifest that if EF=FG, FI will be a mean proportional between EC and GK; and that, if E*f* and *fg* be equal, *fi* will be a mean proportional between EC and *gk*.

23. Corollary. If through *l*, *k*, *i*, C, I, K, L a curve be described, it will be the logarithmic curve.

Fig. 26. 24. Problem. Given the tension at the vertex, to construct an indefinite catenary.

On any indefinite vertical line BD, make BA perpendicular, and equal to the given tension at the vertex, through A draw AC, making the angle A=56°·28′ and meeting BD in C, with the centre C and distance CA, cut BD in D; then, through D, draw DE parallel to BA meeting AC produced in E; produce DE, so that DF=DE and take CV=BA: then, by Case I., article 19., shall E, V, and F, be three points in the proposed catenary; and LCK, drawn through C parallel to BA or ED, will be its directrix. Through E and F draw KEH, FGL parallel to CD, making the parts EH and FG each equal to CB. Bisect CK, CL in N and M, through which points, parallel to CD, draw NO, MP, making the former a mean proportional between CV and KH, and the latter a mean proportional between CV and LG: join OP, cutting the axis CD in Q, and through Q draw RQS parallel to FE, cutting NO in S, and MP produced in R, so shall R and S (by article 21.) be points in the proposed curve. In the same manner, by bisecting CN and CM, and finding a mean proportional between CV and NO, and another between CV and MP, two other points may be found in the catenary. Also by bisecting NK and LM, and finding mean proportionals between NO, KH and MP, LG, two other points in the curve may be found, the one between S and E, and the other between R and F. And by continually bisecting the divisions on LK, and finding mean proportionals as before, as many points as we please may be found between V and E, or between VF.

Again, in order to continue the curve beyond the points E and F, take KT, and LU each equal to any one of the divisions already made on CK, say equal to NK, and through the points T and U draw TI and UW, making the former a third proportional to NO and KH, and the latter a third proportional to MP and LG; join IW cutting the axis in X, and through X draw YXZ parallel to FE, cutting TI in Z and UW produced in Y, so shall Y and Z be two points in the curve, as is evident from article 21. In like manner by taking mean and third proportionals the construction may be carried to any extent, and to any degree of exactness.

25. Problem. Given, any two of the six quantities c, x, t, φ, z, and y, to construct the catenary.

CASE I.

Fig. 27. When c and x are the two given quantities.

By the last Problem, with c, the given tension at the vertex, construct the indefinite catenary ACB, having CD for its axis and EK for its directrix. From the vertex C on the axis CD, lay off CF equal the other given quantity x, and through F draw the ordinate FG, cutting the curve ABC in G, then shall CGF be the catenary required, as is evident from the foregoing articles without any formal demonstration.

CASE II.

When ct, $c\phi$, cz, xt, $x\phi$, xz, $t\phi$, tz, or ϕz are given.

By means of the figure EIFC find c and x by article 20. and then proceed as in Case I. and the thing is done.

CASE III.

When c and y are the two given quantities.

On the directrix EK, of the indefinite catenary ACB, constructed as directed in Case I. lay off EH equal to the given quantity y, and through H draw HG parallel to the axis CD, intersecting the curve ACB in G, and through G draw the ordinate GF and the construction is done.

CASE IV.

When x and y are the two given quantities.

From CE, any assumed tension at the vertex, construct the indefinite catenary ACB as in Case I., and on its axis CD lay off CL equal to the given quantity x, and through L, perpendicular to CL, draw LM equal to the other given quantity y. Through C and M draw the straight line CM, cutting the curve ACB in G, and through G draw the ordinate GF. Take any point N, between C and G, in the curve ACB, and through C and N draw the indefinite straight line CNO; join FN, and through L, parallel to FN, draw LP cutting CO in P, then will P be a point in the required catenary. In the same manner any number of points in the required catenary may be found between C and M; and consequently a curve CPM drawn through C, M, and through all the points so found, will be the catenary required, as is evident from article 13., the figures CGF, CML being similar, and the latter having the given data x and y. Again, if GE be joined, and through M, parallel to GE, MQ be drawn, cutting the axis produced in Q, CQ will represent the tension at the vertex of the catenary CPM, as is evident by article 13. from the similarity of the triangles EFG, QLM.

CASE V.

When t and y are the two given quantities.

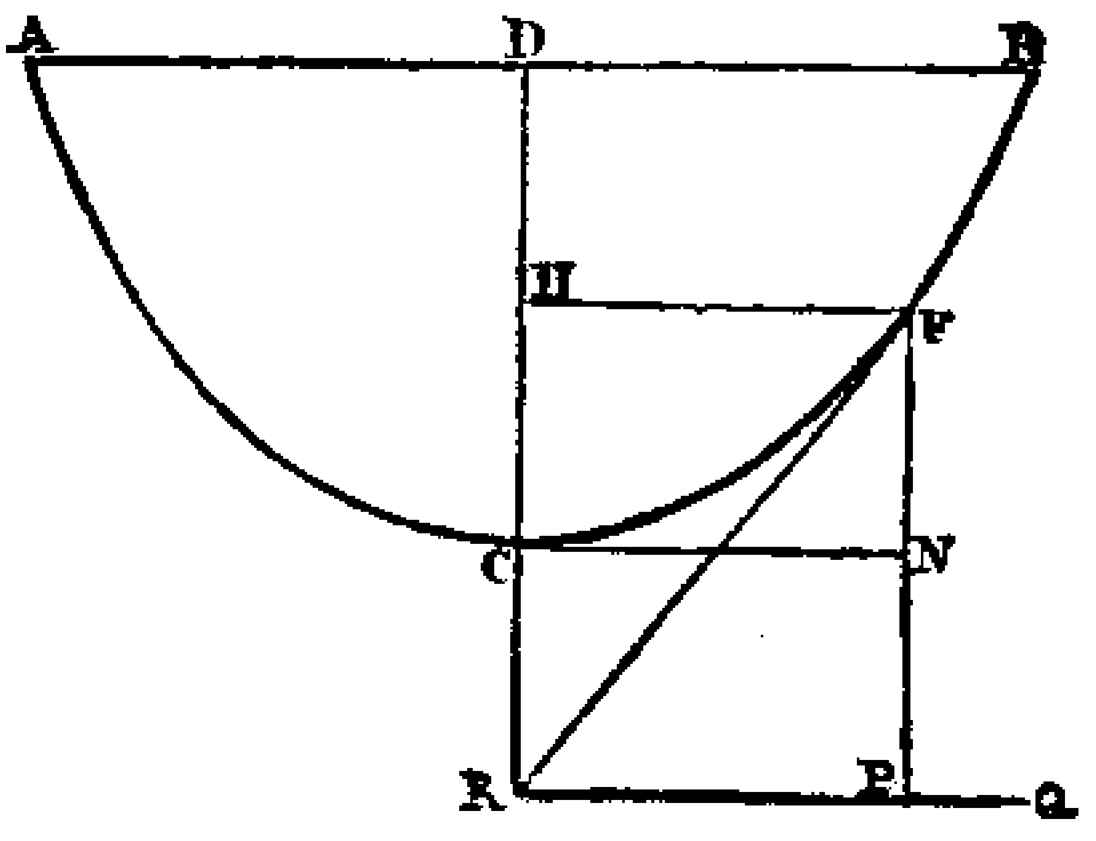
From any assumed tension CE at the vertex, construct the indefinite catenary ACB as before; and on its directrix lay off EK equal to the given quantity y, and through K draw KR parallel to the axis CD, and equal to the given quantity t, and join ER. Then if ER fall without the catenary ACB, the solution is *impossible*, and if ER touch the curve ACB, t is then at its minimum with respect to y, and the problem may in that case be solved by art. 19. and 24. But if ER cut the curve ACB, it must by art. 11. cut it in two points as G and S; and in that case the problem admits of *two* solutions agreeable to the observation made in Case XII. of art. 9. For, through C and G draw CGM cutting KR in M, and through M draw the ordinate ML; and construct the catenary CPM, similar to CNG as in the last case; then shall CPM be *one* of the catenaries required; and by proceeding with the point S, as we have done with G, the *other* catenary having the proposed data, may be constructed. For since EQ=MR, and LE = MK, being opposite sides of parallelograms, by adding equals to equals, we have QL = KR = t and LM = EK=y, and these being proportional to the like quantities EF and FG in the catenary CGF, the problem is evident by art. 13.

CASE VI.

When φ and y are the two given quantities.

With the given quantity φ and any assumed tension at the vertex, construct the catenary CGF as in Case II. On its directrix lay off EK equal to the given quantity y, and through K, parallel to the axis CD, draw the indefinite straight line KR: then through C and G draw CGM cutting KR in M, and through M draw the ordinate ML; then construct the figure CPM similar to CNG as in Case IV., and it will be the catenary required.

26. Problem. To find the area of a catenary.



Through C draw CN parallel to HF, meeting FP in N. Then by Art. 8., we have $t\dot{y} = c\dot{z}$ or by substituting $c+x$ for t, $(c+x)\dot{y} = c\dot{z}$, or $x\dot{y} = c\dot{z} - c\dot{y}$, But $x\dot{y}$ is the fluxion of the external area, CFN, consequently the area CFN $= cz - cy$; hence that of the internal area CFH $=$ CNFH $-$ CFN$= xy - (cz - cy) = cy + xy - cz = \overline{c+x}\,y - cz = ty - cz =$ PF.FH $-$ CR.CF, which was to be found.

27. Problem. To find the radius of curvature to any point in the catenary.

From art. 8., we have $\dot{y} = \frac{c\dot{x}}{z}$ and $\dot{z} = \frac{t\dot{x}}{z}$ or by making $\dot{x} = 1$, $y = \frac{c}{z} = cz^{-1}$ and $\dot{z} = \frac{t}{z}$; hence $-\ddot{y} = \frac{c\dot{z}}{z^2}$ and consequently $\frac{\dot{z}^3}{-\ddot{y}} = \frac{t^3}{z^3} \div \frac{c\dot{z}}{z^2} = \frac{t^3}{z^3} \times \frac{z^2}{c\dot{z}} = \frac{t^3}{z^3} \times \frac{z^2}{c} \times \frac{z}{t} = \frac{t^3z^3}{ctz^3} = \frac{t^2}{c} = \frac{(c+x)^2}{c}$ $=$ the radius of curvature required.

28. Corollary. At the vertex $x = o$, and the radius of curvature $\frac{(c+x)^2}{c}$ in that case becomes $= \frac{c^2}{c} = c =$ the tension at that point.

29. Problem. to find the centre of gravity of the catenarian *curve.*

By the principles of fluxions we have the distance of the centre of gravity of any curve from its vertex, $=\frac{f'x\dot{z}^*}{z}=\frac{1}{z}f'x\dot{z}$.

Now since $\dot{y}=\frac{c\dot{x}}{z}=\frac{c\dot{x}}{\sqrt{2cx+x^2}}$ and

$$\dot{z}=\dot{F}\sqrt{2cx+x^2}=\frac{(c+x)\dot{x}}{\sqrt{2cx+x^2}}=\frac{c\dot{x}}{\sqrt{2cx+x^2}}+\frac{x\dot{x}}{\sqrt{2cx+x^2}}$$

which, by substituting $\dot{y}$ for its equal, becomes $\dot{z}=\dot{y}+\frac{x\dot{x}}{\sqrt{2cx+x^2}}$ or $\frac{x\dot{x}}{\sqrt{2cx+x^2}}=\dot{z}-\dot{y}$ or, by multiplying each side by c, $\frac{cx\dot{x}}{\sqrt{2cx+x^2}}=c\ (\dot{z}-\dot{y})$ and consequently $F'\frac{cx\dot{x}}{\sqrt{2cx+x^2}}=c\ (z-y)$.

Again assume $xz=p$.

Then, $\dot{x}z+x\dot{z}=\dot{p}$ or by substitution $\dot{x}\sqrt{2cx+x^2}+\frac{(c+x)x\dot{x}}{\sqrt{2cx+x^2}}=\frac{\dot{x}(2cx+x^2)+cx\dot{x}+x^2\dot{x}}{\sqrt{2cx+x^2}}=\frac{3cx\dot{x}}{\sqrt{2cx+x^2}}+\frac{2cx^2\dot{x}}{\sqrt{2cx+x^2}}=\dot{p}$ or $\frac{2cx^2\dot{x}}{\sqrt{2cx+x^2}}=\dot{p}-\frac{3cx\dot{x}}{\sqrt{2cx+x^2}}$ or $F'\frac{x^2\dot{x}}{\sqrt{2cx+x^2}}=\frac{p}{2}-\frac{3c}{2}(z-y)$

Whence $F' x\dot{z} = F' \frac{(c+x)x\dot{x}}{\sqrt{2cx+x^2}} = F' \frac{cx\dot{x}}{\sqrt{2cx+x^2}} + F' \frac{x^2\dot{x}}{\sqrt{2cx+x^2}} = c(z-y) + \frac{p}{2} - \frac{3c}{2}(z-y) = \frac{p}{2} - \frac{c}{2}(z-y) = \frac{xz}{2} - \frac{c}{2}(z-y) = \frac{xz+cy-cz}{2}$ and $\frac{1}{z} F' x\dot{z}$ is therefore $= \frac{xz+cy-cz}{2z} = \frac{1}{2}(x+\frac{cy}{z}-c) =$ the distance of the centre of gravity from the vertex, which was to be found.

DIMENSIONS OF A CATENARY.

THE tension at the vertex$=1=c$.

$\varphi=$ the angle which the catenary makes with a horizontal line at each point of suspension.

$x=$ the absciss.

$y=$ the ordinate.

$z=$ half the length of the chain.

$t=$ the measure of the tension of the chain at the point of suspension.

It may be observed that all the dimensions except y may be directly obtained from the table to every five minutes of a degree; and when $m=2.302585=$the number by which the common logarithm of any number must be multiplied to obtain the hyperbolic logarithm of the same number, then y and the other quantities may be obtained at any angle by the following table of formulæ.

Note. The tangent, secant, cosecant, and sine of an angle may be found directly from the table, and it is to be observed that when the angle $=\varphi$, radius $=1$, and arc $=\varphi^\circ$.

The cotan. of $\varphi=$ tangent of $90^\circ-\varphi=\frac{1}{z}$

The cosec. of $\varphi=$ secant of $90^\circ-\varphi=\frac{t}{z}$

The cosine of $\varphi=$ sine of $90^\circ-\varphi=\frac{1}{t}$

The ver.sin. of $\varphi=1-$ cosine of $\varphi=1-\frac{1}{t}$

The covers. of $\varphi=1-$ sine of $\varphi=1-\frac{z}{t}$

Arc. $=.0174533\times\varphi^\circ$.

FORMULÆ to find any four of the six quantities of the common Catenary, two of them being given. Let the six quantities be denoted as follows: φ = the angle; c = the constant quantity; t = the tension; z = the length of the chain between the point of suspension, where φ is, and the apex; x = the absciss; and y = the ordinate; and let $m = 2.302585$, the number by which the common logarithm of any number must be multiplied to obtain the hyperbolic logarithm of the same number.

GIVEN QUANTITIES.	REQUIRED QUANTITIES.					
	φ	c	t	z	y	x
φ and c	- - - - -	- - - -	$c \sec. \phi$	$c \tan g. \phi$	$cm \log. \tan g. (45° + \frac{1}{2}\phi)$	$c (\sec. \phi - 1) = \frac{c \text{ ver. sin. } \phi}{\cos. \phi}$
φ and t	- - - - -	$\frac{t}{\sec. \phi} = t \cos. \phi$	- - - -	$t \sin. \phi$	$mt \cos. \phi \log. \tan g. (45° + \frac{1}{2}\phi)$	$t \text{ ver. sin. } \varphi$
φ and z	- - - - -	$z \cot. \phi$	$z \text{ cosec. } \phi$	- - - -	$mz \cot. \phi \log. \tan g. (45° + \frac{1}{2}\phi)$	$z \text{ cosec. } \varphi \text{ ver. sin. } \phi = \frac{z \text{ ver. sin. } \phi}{\sin. \phi}$
φ and x	- - - - -	$\frac{x}{\sec. \phi - 1} = \frac{x \cos. \phi}{\text{ver. sin. } \phi}$	$\frac{x}{\text{ver. sin. } \varphi}$	$\frac{x \sin. \phi}{\text{ver. sin. } \phi}$	$\frac{mx \cos. \phi \log. \tan g. (45° + \frac{1}{2}\phi)}{\text{ver. sin. } \phi}$	- - - - -
φ and y	- - - - -	$\frac{y}{m \log. \tan g. (45° + \frac{1}{2}\phi)}$	$\frac{y}{m \cos. \phi \log. \tan g. (45° + \frac{1}{2}\phi)}$	$\frac{y \tan g. \phi}{m \log. \tan g. (45° + \frac{1}{2}\phi)}$	- - - -	$\frac{y \text{ ver. sin. } \phi}{m \cos. \phi \log. \tan g. (45° + \frac{1}{2}\phi)}$
c and t	$\sec. \phi = \frac{t}{c}$	- - - -	- - - -	$\sqrt{t^2 - c^2}$	$cm \log. \frac{t + \sqrt{t^2 - c^2}}{c}$	$t - c$
c and z	$\tan g. \varphi = \frac{z}{c}$	- - - -	$\sqrt{z^2 + c^2}$	- - - -	$cm \log. \frac{\sqrt{z^2 + c^2} + z}{c}$	$\sqrt{z^2 + c^2} - c$
c and y	$\log. \tan g. (45° + \frac{1}{2}\varphi) = \frac{y}{cm}$ *	- - - -	$c \sec. \phi$ *	$c \tan g. \phi$ *	- - - -	$c (\sec. \phi - 1) = \frac{{}^{*}c \text{ ver. sin. } \phi}{\cos. \phi}$
c and x	$\sec. \varphi = 1 + \frac{x}{c}$	- - - -	$c + x$	$\sqrt{2 cx + x^2}$	$cm \log. \frac{c + x + \sqrt{2 cx + x^2}}{c}$	- - - - -
t and z	$\sin. \varphi = \frac{z}{t}$	$\sqrt{t^2 - z^2}$	- - - -	- - - -	$m \sqrt{t^2 - z^2} \log. \frac{t + z}{\sqrt{t^2 - z^2}}$	$t - \sqrt{t^2 - z^2}$
t and x	$\sec. \varphi = \frac{t}{t - x}$	$t - x$	- - - -	$\sqrt{2 tx - x^2}$	$m (t - x) \log. \frac{t + \sqrt{2 tx - x^2}}{t - x}$	- - - - -
t and y	$\cos. \varphi . \log. \tan g. (45° + \frac{1}{2}\varphi) = \frac{y}{mt}$ †	$\frac{t}{\sec. \phi} = t \cos. \phi$ *	- - - -	$t \sin. \phi$ *	- - - -	$t \text{ ver. sin. } \phi$ *
z and x	$\sec. \phi = \frac{z^2 + x^2}{z^2 - x^2}$	$\frac{z^2 - x^2}{2 x}$	$\frac{z^2 + x^2}{2 x}$	- - - -	$m \frac{(z^2 - x^2)}{2 x} \log. \frac{z + x}{z - x}$	- - - - -
z and y	$\cot. \varphi \log. \tan g. (45° + \frac{1}{2}\phi) = \frac{y}{mz}$ ‡	$z \text{ cotang. } \phi$ ‡	$z \text{ cosec. } \phi$ ‡	- - - -	- - - -	$z \text{ cosec. } \varphi \text{ ver. sin. } \phi = \frac{z \text{ ver. sin. } \phi}{\sin. \phi .}$ ‡
x and y	$\frac{\log. \tan g. (45° + \frac{1}{2}\varphi)}{\sec. \varphi \text{ ver. sin. } \varphi} = \frac{y}{mx}$	$\frac{x \cos. \phi}{\text{ver. sin. } \phi} = \frac{x}{\sec. \phi - 1.}$	$\frac{x}{\text{ver. sin. } \phi}$	$\frac{x \sin. \phi}{\text{ver. sin. } \phi .}$	- - - -	- - - - -

* c and y being given, ϕ must be obtained as in the first column, whence t, z, and x.

† t and y being given, ϕ may be found by approximation, and it may be observed ϕ in this case has two values, the one above 56° 28′ and the other below that quantity; and consequently c, x and z must have two values.

‡ z and y being given, ϕ may be found by approximation, and c, t, and x from z, ϕ being found as in the first column; or, x and y being given, ϕ may be found by approximation, and c, t, and z from x, ϕ being found as in the first column.

FORMULÆ TO FIND THE SUBTANGENT, THE AREA, THE CENTRE OF GRAVITY, AND THE RADIUS OF CURVATURE.

THE subtangent $= \frac{zy}{c}$ and when t is a minimum in respect to y, that is, when $\phi = 56^\circ\,28'$, the subtangent $= t = c + x$.

The area cut off by a double ordinate $= 2\,(ty - cz)$.

The greatest area under a given *curved* boundary, is when $(2 \operatorname{cosec.}\phi - \sin.\phi) \log.\tan g.\,(45^\circ + \frac{1}{2}\phi) = \frac{2}{m}$ or when $\phi = 67^\circ\,18'$.

The greatest area under a given perimeter (that is, the curve and double ordinate,) is when $\frac{(n+1)\,\tan g.^2\phi}{2n\,(\sec.\phi + 1)\,(\sec.\phi - n)} = 1$, (putting $n = \frac{\tan g.\phi}{m \log.\tan g.\,(45^\circ + \frac{1}{2}\phi)}$) or when $\phi = 79^\circ\,44'$.

The distance of the centre of gravity of the *curve* $2z$ from the vertex $= \frac{1}{2}\left(x + \frac{cy}{z} - c\right)$.

The radius of curvature $= \frac{(c+x)^2}{c} = \frac{t^2}{c}$ which at the vertex $= c$ *; and it varies as the square of the secant of the angle ϕ or as the square of the tension t.

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* When $c = x$, or the radius of curvature at the vertex equals the absciss, $\phi = 60^\circ$, and the catenary will touch at the vertex a semicircle inscribed within it. When the absciss and ordinate are given to draw the catenary, then in column $\frac{y}{x}$, ϕ, x and y are obtained; and if a tangent be drawn (from the angle) at y, of the table, and the semicircle of curvature at the vertex with radius 1, taken from the extremity of x of the table, the hand may draw very accurately the catenary, touching the tangent without, and the semicircle within, the catenary. And it may be observed that c is always less than x when the angle at the ordinate is more than 60°, less than y when the angle is more than $49^\circ\,37'$, and less than z when the angle is more than 45°, and $c = z$ at 45°, $c = y$ at $49\,37'$, $c = x$ at 60, and $x = y$ at $67^\circ\,31'$. It may also be observed that the secant of the angle being the square root of the radius of curvature at that angle, when $c = 1$, by reference to a table of secants and one of square roots, the angle may be taken out, where the radius of curvature is any multiple of c; for instance, at 45° the radius of curvature $= 2\,c$, at $54^\circ\,44' = 3\,c$, at $60^\circ = 4\,c$, at $63^\circ\,26'' = 5\,c$, at $65^\circ\,54' = 6\,c$, at $67^\circ\,48' = 7\,c$, at $69^\circ\,18 = 8\,c$, and at $70^\circ\,32 = 9\,c$.

ϕ	x =sec. of ϕ—1	y	z =tan. of ϕ	t =sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ =cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ =sine of ϕ
deg. min.										
0″ 5	.00000		.00145	1.0000				687.5496		.00145
0″10	.00000		.00291	1.0000				343.7751		.00291
0″15	.00001		.00436	1.0000				229.1838		.00436
0″20	.00002		.00582	1.0000				171.8883		.00582
0″25	.00003		.00727	1.0000				137.5111		.00727
0″30	.00004		.00873	1.0000				114.5930		.00873
0″35	.00005		.01018	1.0000				98.2230		.01018
0″40	.00007		.01164	1.0001				85.9456		.01164
0″45	.00009		.01309	1.0001				76.3966		.01309
0″50	.00011		.01454	1.0001				68.7574		.01454
0″55	.00013		.01600	1.0001				62.5072		.01600
1″ 0	.00015	.01745	.01745	1.0001	114.586	114.584	1.00004	57.2987	57.301	.01745
1″ 5	.00018		.01891	1.0002				52.8916		.01891
1″10	.00021		.02036	1.0002				49.1141		.02036
1″15	.00024		.02182	1.0002				45.8402		.02181
1″20	.00027		.02327	1.0003				42.9757		.02327
1″25	.00030		.02473	1.0003				40.4482		.02472
1″30	.00034		.02618	1.0003				38.2015		.02618
1″35	.00038		.02764	1.0004				36.1914		.02763
1″40	.00042		.02910	1.0004				34.4823		.02908
1″45	.00047		.03055	1.0005				32.7455		.03053
1″50	.00051		.03200	1.0005				31.2576		.03199
1″55	.00056		.03346	1.0006				29.8990		.03344
2″ 0	.00061	.03491	.03492	1.0006	57.279	57.2900	1.00019	28.6537	28.659	.03490
2″ 5	.00066		.03638	1.0007				27.5080		.03635
2″10	.00072	.03782	.03783	1.0007	52.870	52.8821	1.00024	26.4505	26.457	.03780
2″15	.00077		.03929	1.0008				25.4713		.03926
2″20	.00083	.04074	.04075	1.0008	49.090	49.1039	1.00029	24.5621	24.569	.04071
2″25	.00089		.04220	1.0009				23.7156		.04216
2″30	.00095	.04365	.04366	1.0009	45.816	45.8294	1.00033	22.9256	22.933	.04362
2″35	.00102		.04512	1.0010				22.1865		.04507
2″40	.00108	.04656	.04658	1.0011	42.949	42.9641	1.00038	21.4937	21.502	.04652
2″45	.00115		.04803	1.0011				20.8428		.04798
2″50	.00122	.04947	.04949	1.0012	40.419	40.4358	1.00042	20.2303	20.239	.04943
2″55	.00130		.05095	1.0013				19.6527		.05088
3″ 0	.00137	.05238	.05241	1.0014	38.171	38.1885	1.00046	19.1073	19.116	.05234
3″ 5	.00144		.05386	1.0014				18.5914		.05379
3″10	.00153	05530	.05533	1.0015	36.159	36.1776	1.00052	18.1026	18.112	.05524
3″15	.00161		.05678	1.0016				17.6389		.05669
3″20	.00169	.05821	.05824	1.0017	34.348	34.3678	1.00058	17.1984	17.208	.05814
3″25	.00178		.35970	1.0018				16.7794		.05959
3″30	.00187	.06112	.06116	1.0019	32.710	32.7303	1.00063	16.3804	16.391	.06104
3″35	.00196		.06262	1.0020				15.9999		.06250
3″40	.00205	.06404	.06408	1.0021	31.220	31.2416	1.00068	15.6368	15.647	.06395
3″45	.00214		.06554	1.0021				15.2898		.06540
3″50	.00224	.06695	.06700	1.0022	29.860	29.8823	1.00074	14.9579	14.969	.06685
3″55	.00234		.06846	1.0023				14.6401		.06830
4″ 0	.00244	.06987	.06993	1.0024	28.613	28.6363	1.00082	14.3356	14.347	.06976
4″ 5	.00254		.07139	1.0025				14.0435		.07120
4″10	.00265	.07279	.07285	1.0026	27.466	27.4899	1.00087	13.7631	13.775	.07266
4″15	.00275		.07431	1.0028				13.4937		.07411
4″20	.00287	.07570	.07578	1.0029	26.406	26.4316	1.00097	13.2347	13.248	.07556
4″25	.00298		.07724	1.0030				12.9855		.07701
4″30	.00309	.07862	.07870	1.0031	25.425	25.4517	1.00105	12.7455	12.759	.07846

φ	x =sec. of φ	y	z =tan. of φ	t =sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ =cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ =sine of φ
deg. min.										
4″35	.00320		.08016	1.0032				12.5142		.07990
4″40	.00333	.08154	.08163	1.0033	24.515	24.5416	1.00111	12.2913	12.305	.08136
4″45	.00345		.08309	1.0034				12.0761		.08281
4″50	.00357	.08446	.08456	1.0036	23.667	23.6945	1.00118	11.8684	11.882	.08426
4″55	.00369		.08602	1.0037				11.6677		.08570
5″ 0	.00382	.08738	.08749	1.0038	22.874	22.9038	1.00129	11.4737	11.488	.08716
5″ 5	.00395		.08895	1.0039				11.2861		.08860
5″10	.00408	.09030	.09042	1.0041	22.134	22.1640	1.00135	11.1045	11.120	.09005
5″15	.00421		.09189	1.0042				10.9288		.09150
5″20	.00435	.09322	.09335	1.0043	21.439	21.4704	1.00146	10.7585	10.774	.09295
5″25	.00448		.09482	1.0045				10.5935		.09440
5″30	.00463	.09614	.09629	1.0046	20.786	20.8188	1.00155	10.4334	10.450	.09584
5″35	.00477		.09776	1.0048				10.2782		.09729
5″40	.00491	.09906	.09923	1.0049	20.173	20.2056	1.00163	10.1275	10.144	.09874
5″45	.00506		.10069	1.0050				9.9812		.10019
5″50	.00520	.10199	.10216	1.0052	19.593	19.6273	1.00174	9.8391	9.856	.10164
5″55	.00536		.10363	1.0054				9.7010		.10308
6″ 0	.00551	.10491	.10510	1.0055	19.046	19.0811	1.00183	9.5668	9.584	.10453
6″ 5	.00566		.10657	1.0057				9.4362		.10597
6″10	.00582	.10784	.10805	1.0058	18.529	18.5645	1.00194	9.3092	9.327	.10742
6″15	.00598		.10952	1.0060				9.1855		.10887
6″20	.00614	.11076	.11099	1.0061	18.038	18.0750	1.00204	9.0652	9.084	.11031
6″25	.00630		.11247	1.0063				8.9479		.11176
6″30	.00647	.11369	.11394	1.0065	17.573	17.6106	1.00216	8.8337	8.853	.11320
6″35	.00664		.11541	1.0066				8.7223		.11465
6″40	.00681	.11662	.11688	1.0068	17.131	17.1694	1.00227	8.6138	8.633	.11609
6″45	.00698		.11836	1.0070				8.5079		.11754
6″50	.00715	.11955	.11983	1.0072	16.710	16.7496	1.00238	8.4047	8.425	.11898
6″55	.00733		.12130	1.0073				8.3039		.12043
7″ 0	.00751	.12248	.12278	1.0075	16.309	16.3499	1.00255	8.2055	8.226	.12187
7″ 5	.00769		.12426	1.0077				8.1095		.12331
7″10	.00787	.12541	.12574	1.0079	15.927	15.9687	1.00262	8.0156	8.037	.12476
7″15	.00806		.12722	1.0080				7.9240		.12620
7″20	.00825	.12834	.12869	1.0083	15.562	15.6048	1.00275	7.8344	7.856	.12764
7″25	.00844		.13017	1.0084				7.7469		.12908
7″30	.00863	.13128	.13165	1.0086	15.213	15.2570	1.00288	7.6613	7.683	.13053
7″35	.00882		.13313	1.0088				7.5776		.13197
7″40	.00902	.13421	.13461	1.0090	14.880	14.9244	1.00301	7.4957	7.518	.13341
7″45	.00922		.13609	1.0092				7.4156		.13485
7″50	.00942	.13714	.13758	1.0094	14.560	14.6059	1.00315	7.3372	7.360	.13629
7″55	.00962		.13906	1.0096				7.2604		.13773
8″ 0	.00983	.14008	.14054	1.0098	14.254	14.3007	1.00327	7.1853	7.209	.13917
8″ 5	.01004		.14202	1.0100				7.1117		.14061
8″10	.01024	.14302	.14351	1.0102	13.690	14.0079	1.00341	7.0396	7.064	.14205
8″15	.01046		.14499	1.0105				6.9690		.14349
8″20	.01067	.14596	.14648	1.0107	13.678	13.7267	1.00356	6.8998	6.924	.14493
8″25	.01089		.14796	1.0109				6.8320		.14637
8″30	.01111	.14890	.14945	1.0111	13.407	13.4566	1.00371	6.7655	6.791	.14781
8″35	.01133		.15094	1.0113				6.7003		.14925
8″40	.01155	.15184	.15243	1.0116	13.146	13.1969	1.00385	6.6363	6.662	.15069
8″45	.01178		.15391	1.0118				6.5736		.15212
8″50	.01200	.15478	.15540	1.0120	12.895	12.9469	1.00400	6.5121	6.538	.15356
8″55	.01223		.15689	1.0122				6.4517		.15500
9″ 0	.01247	.15773	.15838	1.0125	12.654	12.7062	1.00415	6.3925	6.419	.15643

ϕ	x =sec. of ϕ−1	y	z =tan. of ϕ	t =sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{y}{z}$	$\frac{t}{z}$ =cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ =sine of ϕ
deg.min.										
9″ 5	.01270		.15988	1.0127				6.3343		.15787
9″10	.01294	.16067	.16137	1.0129	12.421	12.4742	1.00431	6.2772	6.304	.15931
9″15	.01317		.16286	1.0132				6.2211		.16074
9″20	.01342	.16362	.16435	1.0134	12.196	12.2505	1.00447	6.1661	6.194	.16218
9″25	.01366		.16584	1.0137				6.1120		.16361
9″30	.01391	.16657	.16734	1.0139	11.979	12.0346	1.00463	6.0589	6.087	.16505
9″35	.01415		.16884	1.0141				6.0067		.16648
9″40	.01440	.16952	.17033	1.0144	11.770	11.8262	1.00480	5.9554	5.984	.16792
9″45	.01466		.17183	1.0147				5.9049		.16935
9″50	.01491	.17247	.17333	1.0149	11.567	11.6248	1.00497	5.8554	5.885	.17078
9″55	.01517		.17483	1.0152				5.8067		.17222
10″ 0	.01543	.17542	.17633	1.0154	11.372	11.4301	1.00514	5.7588	5.788	.17365
10″ 5	.01569		.17783	1.0157				5.7117		.17508
10″10	.01595		.17933	1.0160				5.6653		.17651
10″15	.01622		.18083	1.0162				5.6197		.17794
10″20	.01649	.18134	.18233	1.0165	10.999	11.0594	1.00549	5.5749	5.606	.17937
10″25	.01675		.18384	1.0168				5.5308		.18080
10″30	.01703		.18534	1.0170				5.4874		.18223
10″35	.01731		.18684	1.0173				5.4447		.18366
10″40	.01758	.18725	.18835	1.0176	10.650	10.7119	1.00585	5.4026	5.434	.18509
10″45	.01786		.18986	1.0179				5.3612		.18652
10″50	.01815		.19136	1.0181				5.3205		.18795
10″55	.01843		.19287	1.0184				5.2803		.18938
11″ 0	.01872	.19318	.19438	1.0187	10.320	10.3854	1.00624	5.2408	5.273	.19081
11″ 5	.01901		.19589	1.0190				5.2019		.19224
11″10	.01930		.19740	1.0193				5.1636		.19366
11″15	.01960		.19891	1.0196				5.1258		.19509
11″20	.01989	.19911	.20042	1.0199	10.001	10.0780	1.00662	5.0886	5.122	.19652
11″25	.02019		.20193	1.0202				5.0520		.19794
11″30	.02049		.20345	1.0205				5.0159		.19937
11″35	.02079		.20497	1.0208				4.9803		.20079
11″40	.02110	.20504	.20648	1.0211	9.720	9.7882	1.00703	4.9452	4.980	.20222
11″45	.02140		.20800	1.0214				4.9106		.20364
11″50	.02171		.20952	1.0217				4.8765		.20507
11″55	.02203		.21104	1.0220				4.8429		.20649
12″ 0	.02234	.21099	.21256	1.0223	9.444	9.5144	1.00743	4.8097	4.845	.20791
12″ 5	.02266		.21408	1.0227				4.7771		.20933
12″10	.02298		.21560	1.0230				4.7448		.21076
12″15	.02330		.21712	1.0233				4.7130		.21218
12″20	.02362	.21694	.21864	1.0236	9.183	9.2553	1.00787	4.6817	4.718	.21360
12″25	.02395		.22017	1.0240				4.6507		.21502
12″30	.02428		.22169	1.0243				4.6202		.21644
12″35	.02461		.22322	1.0246				4.5901		.21786
12″40	.02494	.22290	.22475	1.0249	8.936	9.0098	1.00831	4.5604	4.598	.21928
12″45	.02528		.22628	1.0253				4.5311		.22070
12″50	.02562		.22781	1.0256				4.5022		.22211
12″55	.02596		.22934	1.0260				4.4736		.22353
13″ 0	.02630	.22887	.23087	1.0263	8.701	8.7769	1.00875	4.4454	4.484	.22495
13″ 5	.02665		.23240	1.0266				4.4176		.22637
13″10	.02700		.23393	1.0270				4.3901		.22778
13″15	.02735		.23547	1.0274				4.3630		.22920
13″20	.02770	.23484	.23700	1.0777	8.477	8.5555	1.00921	4.3362	4.376	.23061
13″25	.02806		.23854	1.0281				4.3098		.23203
13″30	.02842		.24008	1.0284				4.2837		.23344

ϕ	x = sec. of ϕ−1	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
deg. min.										
13„35	.02878		.24162	1.0288				4.2579		.23486
13„40	.02914	.24082	.24316	1.0291	8.265	8.3450	1.00969	4.2324	4.273	.23627
13„45	.02950		.24470	1.0295				4.2072		.23769
13„50	.02987		.24624	1.0299				4.1824		.23910
13„55	.03024		.24778	1.0302				4.1578		.24051
14„ 0	.03061	.24681	.24933	1.0306	8.062	8.1443	1.01018	4.1336	4.176	.24192
14„ 5	.03099		.25087	1.0310				4.1096		.24333
14„10	.03137		.25242	1.0314				4.0859		.24474
14„15	.03174		.25397	1.0317				4.0625		.24615
14„20	.03213	.25281	.25552	1.0321	7.869	7.9530	1.01069	4.0394	4.083	.24756
14„25	.03251		.25707	1.0325				4.0165		.24897
14„30	.03290		.25862	1.0329				3.9939		.25038
14„35	.03329		.26017	1.0333				3.9716		.25179
14„40	.03368	.25883	.26172	1.0337	7.684	7.7704	1.01119	3.9495	3.994	.25320
14„45	.03408		.26328	1.0341				3.9277		.25460
14„50	.03447		.26483	1.0345				3.9061		.25601
14„53	.03471	.26273	.26577	1.0347	7.569	7.6560	1.01155	3.8933	3.938	.25685
14„55	.03487		.26640	1.0349				3.8848		.25741
15„ 0	.03528	.26484	.26795	1.0353	7.508	7.5958	1.01173	3.8637	3.909	.25881
15„ 5	.03568		.26951	1.0357				3.8428		.26022
15„10	.03609		.27107	1.0361				3.8222		.26163
15„15	.03650		.27263	1.0365				3.8018		.26303
15„20	.03691	.27087	.27419	1.0369	7.339	7.4287	1.01228	3.7816	3.828	.26443
15„25	.03732		.27576	1.0373				3.7617		.26584
15„30	.03774		.27732	1.0377				3.7420		.26724
15„35	.03816		.27889	1.0382				3.7225		.26864
15„40	.03858	.27691	.28046	1.0386	7.177	7.2687	1.01282	3.7032	3.751	.27004
15„45	.03901		.28203	1.0390				3.6840		.27144
15„50	.03944		.28360	1.0394				3.6652		.27284
15„55	.03987		.28517	1.0399				3.6465		.27424
16„ 0	.04030	.28296	.28675	1.0403	7.021	7.1154	1.01339	3.6280	3.677	.27564
16„ 5	.04073		.28832	1.0407				3.6097		.27704
16„10	.04117		.28990	1.0412				3.5915		.27843
16„15	.04161		.29147	1.0416				3.5736		.27983
16„20	.04206	.28901	.29305	1.0421	6.872	6.9682	1.01398	3.5559	3.606	.28123
16„25	.04250		.29463	1.0425				3.5383		.28262
16„30	.04295		.29621	1.0430				3.5209		.28402
16„35	.04340		.29780	1.0434				3.5037		.28541
16„40	.04385	.29512	.29938	1.0439	6.730	6.8269	1.01458	3.4867	3.537	.28680
16„45	.04431		.30097	1.0443				3.4699		.28820
16„50	.04477		.30255	1.0448				3.4532		.28959
16„55	.04523		.30414	1.0452				3.4366		.29098
17„ 0	.04569	.30116	.30573	1.0457	6.591	6.6912	1.01519	3.4203	3.472	.29237
17„ 5	.04616		.30732	1.0462				3.4041		.29376
17„10	.04663		.30891	1.0466				3.3881		.29515
17„15	.04710		.31051	1.0471				3.3722		.29654
17„20	.04757	.30725	.31210	1.0476	6.459	6.5606	1.01580	3.3565	3.409	.29793
17„25	.04805		.31370	1.0481				3.3409		.29932
17„30	.04853		.31530	1.0485				3.3255		.30070
17„35	.04901		.31690	1.0490				3.3102		.30209
17„40	.04950	.31334	.31850	1.0495	6.331	6.4348	1.0165	3.2951	3.349	.30348
17„45	.04998		.32010	1.0500				3.2801		.30486
17„50	.05047		.32171	1.0505				3.2653		.30625
17„55	.05097		.32331	1.0510				3.2506		.30763

ϕ	x = sec. of $\phi - 1$	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
deg. min.										
18„ 0	.05146	.31946	.32492	1.0515	6.208	6.3138	1.0171	3.2361	3.291	.30902
18„ 5	.05196		.32653	1.0520				3.2217		.31040
18„10	.05246		.32814	1.0525				3.2074		.31178
18„15	.05297		.32975	1.0530				3.1932		.31316
18„20	.05347	.32559	.33136	1.0535	6.080	6.1970	1.0177	3.1792	3.236	.31454
18„25	.05398		.33298	1.0540				3.1653		.31592
18„30	.05449		.33460	1.0545				3.1515		.31730
18„35	.05501		.33621	1.0550				3.1379		.31868
18„40	.05552	.33171	.33783	1.0556	5.974	6.0844	1.0184	3.1244	3.182	.32006
18„45	.05604		.33945	1.0561				3.1110		.32144
18„50	.05657		.34107	1.0566				3.0977		.32282
18„55	.05709		.34270	1.0571				3.0846		.32419
19„ 0	.05762	.33786	.34433	1.0576	5.863	5.9758	1.0191	3.0716	3.130	.32557
19„ 5	.05815		.34596	1.0582				3.0586		.32694
19„10	.05869		.34758	1.0587				3.0458		.32832
19„15	.05922		.34922	1.0592				3.0331		.32969
19„20	.05976	.34403	.35085	1.0598	5.757	5.8708	1.0198	3.0206	3.080	.33106
19„25	.06030		.35248	1.0603				3.0081		.33243
19„30	.06085		.35412	1.0608				2.9957		.33381
19„35	.06140		.35576	1.0614				2.9835		.33518
19„40	.06195	.35020	.35740	1.0619	5.653	5.7694	1.0205	2.9713	3.032	.33655
19„45	.06250		.35904	1.0625				2.9593		.33792
19„50	.06306		.36068	1.0631				2.9474		.33929
19„55	.06362		.36232	1.0636				2.9355		.34065
20„ 0	.06418	.35637	.36397	1.0642	5.553	5.6713	1.0213	2.9238	2.986	.34202
20„ 5	.06474		.36562	1.0647				2.9122		.34339
20„10	.06531		.36727	1.0653				2.9006		.34475
20„15	.06588		.36892	1.0659				2.8892		.34612
20„20	.06645	.36256	.37057	1.0664	5.456	5.5764	1.0221	2.8778	2.941	.34748
20„25	.06703		.37223	1.0670				2.8666		.34884
20„30	.06761		.37388	1.0676				2.8554		.35021
20„35	.06819		.37554	1.0682				2.8444		.35157
20„40	.06878	.36878	.37720	1.0688	5.362	5.4845	1.0228	2.8334	2.898	.35293
20„45	.06936		.37887	1.0694				2.8225		.35429
20„50	.06995		.38053	1.0700				2.8117		.35565
20„55	.07055		.38220	1.0706				2.8010		.35701
21„ 0	.07114	.37502	.38386	1.0711	5.271	5.3955	1.0236	2.7904	2.856	.35837
21„ 5	.07174		.38553	1.0717				2.7800		.35972
21„10	.07235		.38721	1.0723				2.7695		.36108
21„15	.07295		.38888	1.0730				2.7591		.36244
21„20	.07356	.38126	.39055	1.0736	5.183	5.3093	1.0244	2.7488	2.816	.36379
21„25	.07417		.39223	1.0742				2.7386		.36515
21„30	.07479		.39391	1.0748				2.7285		.36650
21„35	.07540		.39560	1.0754				2.7185		.36785
21„40	.07602	.38750	.39727	1.0760	5.097	5.2257	1.0252	2.7085	2.777	.36921
21„45	.07665		.39896	1.0766				2.6986		.37056
21„50	.07727		.40065	1.0773				2.6888		.37191
21„55	.07790		.40233	1.0779				2.6791		.37326
22„ 0	.07853	.39376	.40403	1.0786	5.014	5.1446	1.0261	2.6695	2.739	.37461
22„ 5	.07917		.40572	1.0792				2.6599		.37595
22„10	.07981		.40741	1.0798				2.6504		.37730
22„15	.08045		.40911	1.0804				2.6410		.37865
22„20	.08109	.40005	.41081	1.0811	4.933	5.0658	1.0269	2.6316	2.702	.37999
22„25	.08174		.41251	1.0817				2.6223		.38134

φ	x =sec. ofφ−1	y	z =tan. of φ	t =sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ =cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ =sine of φ
deg.min.										
22„30	.08239		.41421	1.0824				2.6131		.38268
22„35	.08305		.41592	1.0830				2.6040		.38403
22„40	.08370	.40636	.41763	1.0837	4.855	4.9894	1.0277	2.5949	2.667	.38537
22„45	.08436		.41934	1.0844				2.5859		.38671
22„50	.08503		.42105	1.0850				2.5770		.38805
22„55	.08569		.42276	1.0857				2.5681		.38939
23„ 0	.08636	.41267	.42447	1.0864	4.778	4.9152	1.0286	2.5593	2.633	.39073
23„ 5	.08703		.42619	1.0870				2.5506		.39207
23„10	.08771		.42791	1.0877				2.5419		.39341
23„15	.08839		.42963	1.0884				2.5333		.39474
23„20	.08907	.41900	.43136	1.0891	4.704	4.8430	1.0295	2.5247	2.599	.39608
23„25	.08975		.43308	1.0897				2.5163		.39742
23„30	.09044		.43481	1.0904				2.5078		.39875
23„35	.09113		.43654	1.0911				2.4995		.40008
23„40	.09183	.42533	.43828	1.0918	4.632	4.7729	1.0304	2.4912	2.567	.40142
23„45	.09252		.44001	1.0925				2.4829		.40275
23„50	.09322		.44175	1.0932				2.4748		.40408
23„55	.09393		.44349	1.0939				2.4666		.40541
24„ 0	.09464	.43169	.44523	1.0946	4.562	4.7046	1.0314	2.4586	2.536	.40674
24„ 5	.09535		.44697	1.0953				2.4506		.40806
24„10	.09606		.44872	1.0961				2.4426		.40939
24„15	.09678		.45047	1.0968				2.4347		.41072
24„20	.09750	.43807	.45222	1.0975	4.493	4.6382	1.0323	2.4269	2.505	.41204
24„25	.09822		.45397	1.0982				2.4191		.41337
24„30	.09895		.45573	1.0989				2.4114		.41469
24„35	.09968		.45748	1.0997				2.4037		.41602
24„40	.10041	.44447	.45924	1.1004	4.426	4.5736	1.0333	2.3961	2.476	.41734
24„45	.10115		.46101	1.1011				2.3886		.41866
24„50	.10189		.46277	1.1019				2.3811		.41998
24„55	.10263		.46454	1.1026				2.3736		.42130
25„ 0	.10338	.45087	.46631	1.1034	4.361	4.5107	1.0342	2.3662	2.447	.42262
25„ 5	.10413		.46808	1.1041				2.3588		.42394
25„10	.10488		.46985	1.1049				2.3515		.42525
25„15	.10564		.47163	1.1056				2.3443		.42657
25„20	.10640	.45729	.47341	1.1064	4.298	4.4494	1.0352	2.3371	2.419	.42788
25„25	.10716		.47519	1.1072				2.3299		.42920
25„30	.10793		.47697	1.1079				2.3228		.43051
25„35	.10870		.47876	1.1087				2.3158		.43182
25„40	.10947	.46374	.48055	1.1095	4.236	4.3897	1.0363	2.3088	2.392	.43313
25„45	.11025		.48234	1.1102				2.3018		.43444
25„50	.11103		.48414	1.1110				2.2949		.43575
25„55	.11181		.48593	1.1118				2.2880		.43706
26„ 0	.11260	.47021	.48773	1.1126	4.176	4.3315	1.0373	2.2812	2.366	.43837
26„ 5	.11339		.48953	1.1134				2.2744		.43968
26„10	.11419		.49134	1.1142				2.2677		.44098
26„15	.11498		.49314	1.1150				2.2610		.44229
26„20	.11579	.47670	.49495	1.1158	4.117	4.2747	1.0383	2.2543	2.341	.44359
26„25	.11659		.49677	1.1166				2.2477		.44490
26„30	.11740		.49858	1.1174				2.2412		.44620
26„35	.11821		.50040	1.1182				2.2346		.44750
26„40	.11903	.48320	.50222	1.1190	4.060	4.2193	1.0394	2.2282	2.316	.44880
26„45	.11985		.50404	1.1198				2.2217		.45010
26„50	.12067		.50587	1.1207				2.2153		.45140
26„55	.12149		.50769	1.1215				2.2090		.45269

ϕ deg. min.	x = sec. of $\phi-1$	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
27„ 0	.12233	.48971	.50952	1.1223	4.003	4.1653	1.0405	2.2027	2.292	.45399
27„ 5	.12316		.51136	1.1231				2.1964		.45529
27„10	.12400		.51319	1.1240				2.1902		.45658
27„15	.12484		.51503	1.1248				2.1840		.45787
27„20	.12568	.49625	.51688	1.1257	3.948	4.1126	1.0416	2.1779	2.268	.45917
27„25	.12653		.51872	1.1265				2.1717		.46046
27„30	.12738		.52057	1.1274				2.1657		.46175
27„35	.12824		.52242	1.1282				2.1596		.46304
27„40	.12910	.50282	.52427	1.1291	3.895	4.0611	1.0427	2.1536	2.246	.46433
27„45	.12996		.52612	1.1300				2.1477		.46561
27„50	.13082		.52798	1.1308				2.1418		.46690
27„55	.13170		.52984	1.1317				2.1359		.46819
28„ 0	.13257	.50940	.53171	1.1326	3.843	4.0108	1.0438	2.1301	2.223	.46947
28„ 5	.13345		.53358	1.1334				2.1242		.47076
28„10	.13433		.53545	1.1343				2.1185		.47204
28„15	.13521		.53732	1.1352				2.1127		.47332
28„20	.13610	.51599	.53920	1.1361	3.791	3.9617	1.0450	2.1070	2.202	.47460
28„25	.13700		.54107	1.1370				2.1014		.47588
28„30	.13789		.54296	1.1379				2.0957		.47716
28„35	.13879		.54484	1.1388				2.0901		.47844
28„40	.13970	.52262	.54673	1.1397	3.741	3.9136	1.0461	2.0846	2.181	.47971
28„45	.14061		.54862	1.1406				2.0790		.48099
28„50	.14152		.55051	1.1415				2.0736		.48226
28„55	.14243		.55241	1.1424				2.0681		.48354
29„ 0	.14335	.52925	.55431	1.1433	3.692	3.8667	1.0473	2.0627	2.160	.48481
29„ 5	.14428		.55621	1.1443				2.0573		.48608
29„10	.14520		.55812	1.1452				2.0519		.48735
29„15	.14614		.56003	1.1461				2.0466		.48862
29„20	.14707	.53590	.56194	1.1471	3.644	3.8208	1.0486	2.0413	2.140	.48990
29„25	.14801		.56385	1.1480				2.0360		.49116
29„30	.14895		.56577	1.1489				2.0308		.49242
29„35	.14990		.56759	1.1499				2.0256		.49369
29„40	.15085	.54260	.56962	1.1508	3.597	3.7760	1.0498	2.0204	2.121	.49495
29„45	.15181		.57154	1.1518				2.0152		.49622
29„50	.15277		.57348	1.1528				2.0101		.49748
29„55	.15373		.57541	1.1537				2.0050		.49874
30„ 0	.15470	.54930	.57735	1.1547	3.551	3.7321	1.0511	2.0000	2.102	.50000
30„ 5	.15567		.57929	1.1557				1.9950		.50126
30„10	.15665		.58123	1.1566				1.9900		.50252
30„15	.15763		.58318	1.1576				1.9850		.50377
30„20	.15861	.55603	.58513	1.1586	3.506	3.6891	1.0523	1.9801	2.084	.50503
30„25	.16000		.58709	1.1600				1.9752		.50628
30„30	.16059		.58901	1.1606				1.9703		.50754
30„35	.16158		.59100	1.1616				1.9654		.50879
30„40	.16259	.56280	.59297	1.1626	3.461	3.6470	1.0536	1.9606	2.066	.51004
30„45	.16359		.59494	1.1636				1.9558		.51129
30„50	.16460		.59691	1.1646				1.9511		.51254
30„55	.16562		.59888	1.1656				1.9463		.51379
31„ 0	.16663	.56957	.60086	1.1666	3.418	3.6059	1.0549	1.9416	2.048	.51504
31„ 5	.16765		.60284	1.1676				1.9369		.51628
31„10	.16868		.60483	1.1687				1.9323		.51753
31„17	.17012	.5751	.60761	1.1701	3.383	3.573	1.056	1.9258	2.034	.51927
31„20	.17075		.60881	1.1707				1.9230		.52002
31„25	.17178		.61080	1.1718				1.9184		.52126

φ	x = sec. of φ	y	z = tan of φ	t = sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of φ
deg. min.										
31„30	.17283		.61280	1.1728				1.9139		.52250
31„35	.17387		.61480	1.1739				1.9093		.52374
31„40	.17493		.61681	1.1749				1.9048		.52498
31„45	.17598		.61882	1.1760				1.9004		.52621
31„50	.17704		.62083	1.1770				1.8959		.52745
31„55	.17811		.62285	1.1781				1.8915		.52868
32„ 0	.17918		.62487	1.1792				1.8871		.52992
32„ 4	.18004	.5912	.62649	1.1800	3.284	3.480	1.060	1.8836	1.996	.53091
32„10	.18133		.62892	1.1813				1.8783		.53238
32„15	.18241		.63095	1.1824				1.8740		.53361
32„20	.18350		.63299	1.1835				1.8669		.53484
32„25	.18459		.63503	1.1846				1.8654		.53607
32„30	.18569		.63707	1.1857				1.8611		.53730
32„35	.18679		.63912	1.1868				1.8569		.53852
32„40	.18789		.64117	1.1879				1.8527		.53975
32„45	.18900		.64322	1.1890				1.8485		.54097
32„50	.19012	.6070	.64528	1.1901	3.195	3.395	1.063	1.8443	1.960	.54220
32„55	.19124		.64734	1.1912				1.8402		.54342
33„ 0	.19236		.64941	1.1924				1.8361		.54464
33„ 5	.19349		.65148	1.1935				1.8320		.54586
33„10	.19462		.65355	1.1946				1.8279		.54708
33„15	.19576		.65563	1.1958				1.8238		.54829
33„20	.19691		.65771	1.1969				1.8198		.54951
33„25	.19805		.65980	1.1980				1.8158		.55072
33„30	.19920		.66188	1.1992				1.8118		.55194
33„33	.20000	.6223	.66314	1.2000	3.111	3.316	1.066	1.8094	1.928	.55266
33„35	.20036		.66400	1.2004				1.8078		.55315
33„40	.20152		.66608	1.2015				1.8039		.55436
33„45	.20269		.66818	1.2027				1.7999		.55557
33„50	.20386		.67028	1.2039				1.7960		.55678
33„55	.20504		.67239	1.2050				1.7922		.55799
34„ 0	.20622		.67451	1.2062				1.7883		.55919
34„ 5	.20740		.67663	1.2074				1.7844		.56040
34„10	.20859		.67875	1.2085				1.7806		.56160
34„16	.21003	.6371	.68130	1.2100	3.034	3.244	1.069	1.7761	1.899	.56304
34„20	.21099		.68301	1.2110				1.7730		.56401
34„25	.21220		.68514	1.2122				1.7693		.56521
34„30	.21341		.68728	1.2134				1.7655		.56641
34„35	.21462		.68942	1.2146				1.7618		.56760
34„40	.21584		.69157	1.2158				1.7581		.56880
34„45	.21707		.69372	1.2171				1.7544		.57000
34„50	.21830		.69588	1.2183				1.7507		.57119
34„55	.21953		.69804	1.2195				1.7471		.57238
34„57	.22003	.6518	.69891	1.2200	2.963	3.177	1.072	1.7456	1.872	.57286
35„ 0	.22077		.70021	1.2208				1.7434		.57358
35„ 5	.22202		.70238	1.2202				1.7398		.57477
35„10	.22327		.70455	1.2233				1.7362		.57596
35„15	.22453		.70673	1.2245				1.7327		.57714
35„20	.22579		.70891	1.2258				1.7291		.57833
35„25	.22705		.71110	1.2270				1.7256		.57952
35„30	.22833		.71329	1.2283				1.7220		.58070
35„35	.22960		.71549	1.2296				1.7185		.58189
35„37	.23012	.6658	.71637	1.2301	2.895	3.114	1.076	1.7171	1.847	.58236
35„40	.23089		.71769	1.2309				1.7151		.58307

φ	x = sec. of φ − 1	y	z = tan. of φ	t = sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of φ
deg. min.										
35„45	.23217		.71990	1.2321				1.7116		.58425
35„50	.23347		.72211	1.2335				1.7081		.58543
35„55	.23476		.72432	1.2348				1.7047		.58661
36„ 0	.23607		.72654	1.2361				1.7013		.58778
36„ 5	.23738		.72877	1.2374				1.6979		.58896
36„10	.23869		.73100	1.2387				1.6945		.59014
36„15	.24001	.6796	.73323	1.2400	2.832	3.055	1.079	1.6912	1.825	.59131
36„20	.24134		.73547	1.2413				1.6878		.59248
36„25	.24267		.73771	1.2427				1.6845		.59365
36„30	.24400		.73996	1.2440				1.6812		.59482
36„35	.24534		.74221	1.2453				1.6779		.59599
36„40	.24669		.74447	1.2467				1.6746		.59716
36„45	.24804		.74673	1.2480				1.6713		.59832
36„52	.24995	.6932	.74991	1.2499	2.773	3.000	1.082	1.6668	1.803	.59995
36„55	.25077		.75128	1.2508				1.6649		.60065
37„ 0	.25214		.75355	1.2521				1.6616		.60181
37„ 5	.25351		.75584	1.2535				1.6584		.60298
37„10	.25489		.75812	1.2549				1.6553		.60414
37„15	.25628		.76042	1.2563				1.6521		.60529
37„20	.25767		.76271	1.2577				1.6489		.60645
37„25	.25907		.76502	1.2591				1.6458		.60761
37„28	.25991	.7063	.76640	1.2599	2.717	2.948	1.085	1.6439	1.784	.60830
37„30	.26047		.76733	1.2605				1.6427		.60876
37„35	.26188		76964	1.2619				1.6396		.60991
37„40	.26330		.77196	1.2633				1.6365		.61107
37„45	.26472		.77428	1.2647				1.6334		.61222
37„50	.26615		.77661	1.2661				1.6303		.61337
37„55	.26758		.77895	1.2676				1.6273		.61451
38„ 0	.26902		.78128	1.2690				1.6243		.61566
38„ 3	.26988	.7192	.78269	1.2699	2.664	2.900	1.089	1.6225	1.766	.61635
38„ 5	.27046		.78363	1.2705				1.6213		.61681
38„10	.27191		.78598	1.2719				1.6183		.61795
38„15	.27337		.78834	1.2734				1.6153		.61909
38„20	.27483		.79070	1.2748				1.6123		.62024
38„25	.27630		.79306	1.2763				1.6093		.62138
38„30	.27778		.79544	1.2778				1.6064		.62251
38„35	.27926		.79781	1.2793				1.6035		.62365
38„37	.27985	.7318	.79877	1.2798	2.614	2.854	1.092	1.6023	1.749	.62411
38„40	.28075		.80020	1.2807				1.6005		.62479
38„45	.28224		.80258	1.2822				1.5976		.62592
38„50	.28374		.80498	1.2837				1.5947		.62706
38„55	.28525		.80738	1.2852				1.5919		.62819
39„ 0	.28676		.80978	1.2868				1.5890		.62932
39„ 5	.28828		.81220	1.2883				1.5862		.63045
39„11	.29011	.7443	.81510	1.2901	2.567	2.810	1.095	1.5828	1.733	.63180
39„15	.29133		.81703	1.2913				1.5805		.63271
39„20	.29287		.81946	1.2929				1.5777		.63383
39„25	.29442		.82190	1.2944				1.5749		.63495
39„30	.29597		.82434	1.2960				1.5721		.63608
39„35	.29752		.82678	1.2975				1.5694		.63720
39„40	.29909		.82923	1.2991				1.5666		.63832
39„43	.30003	.7564	.83071	1.3000	2.521	2.769	1.098	1.5650	1.719	.63899
39„45	.30066		.83169	1.3007				1.5639		.63944
39„50	.30223		.83415	1.3022				1.5611		.64056

ϕ	x = sec. of $\phi-1$	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
deg.min.										
39„55	.30382		.83662	1.3038				1.5584		.64167
40„ 0	.30541		.83910	1.3054				1.5557		.64279
40„ 5	.30700		.84158	1.3070				1.5530		.64390
40„10	.30861		.84407	1.3086				1.5504		.64501
40„14	.30989	.7684	.84606	1.3099	2.479	2.730	1.101	1.5482	1.705	.64590
40„20	.31183		.84906	1.3118				1.5450		.64723
40„25	.31346		.85156	1.3135				1.5424		.64834
40„30	.31509		.85408	1.3151				1.5398		.64945
40„35	.31672		.85659	1.3167				1.5372		.65056
40„40	.31837		.85912	1.3184				1.5345		.65166
40„45	.32002	.7801	.86165	1.3200	2.438	2.692	1.104	1.5320	1.692	.65276
40„50	.32168		.86419	1.3217				1.5294		.65386
40„55	.32334		.86673	1.3233				1.5268		.65496
41„ 0	.32501		.86929	1.3250				1.5243		.65606
41„ 5	.32669		.87184	1.3267				1.5217		.65716
41„10	.32838		.87440	1.3284				1.5192		.65825
41„15	.33007	.7915	.87698	1.3301	2.398	2.657	1.108	1.5167	1.680	.65935
41„20	.33177		.87955	1.3318				1.5141		.66044
41„25	.33348		.88213	1.3335				1.5116		.66153
41„30	.33519		.88472	1.3352				1.5092		.66262
41„35	.33691		.88732	1.3369				1.5067		.66371
41„40	.33864		.88992	1.3386				1.5042		.66480
41„44	.34004	.8029	.89201	1.3400	2.362	2.624	1.111	1.5023	1.669	.66566
41„50	.34212		.89515	1.3421				1.4993		.66670
41„55	.34387		.89778	1.3439				1.4969		.66805
42„ 0	.34563		.90040	1.3456				1.4945		.66913
42„ 5	.34740		.90304	1.3474				1.4921		.67021
42„10	.34917		.90568	1.3492				1.4897		.67129
42„12	.34988	.8140	.90674	1.3499	2.326	2.591	1.114	1.4887	1.659	.67172
42„15	.35095		.90834	1.3509				1.4873		.67237
42„20	.35274		.91099	1.3527				1.4849		.67344
42„25	.35454		.91366	1.3545				1.4825		.67452
42„30	.35634		.91633	1.3563				1.4802		.67559
42„35	.35815		.91901	1.3581				1.4778		.67666
42„40	.35997	.8249	.92170	1.3600	2.291	2.560	1.117	1.4755	1.649	.67773
42„44	.36143		.92385	1.3614				1.4736		.67859
42„50	.36363		.92709	1.3636				1.4709		.67987
42„55	.36548		.92980	1.3655				1.4686		.68093
43„ 0	.36733		.93251	1.3673				1.4663		.68200
43„ 5	.36919		.93524	1.3692				1.4640		.68306
43„ 7	.36993	.8357	.93633	1.3700	2.258	2.531	1.120	1.4631	1.639	.68348
43„10	.37105		.93797	1.3710				1.4617		.68412
43„15	.37293		.94071	1.3729				1.4595		.68518
43„20	.37481		.94341	1.3748				1.4572		.68624
43„25	.37670		.94620	1.3767				1.4550		.68730
43„30	.37860		.94896	1.3786				1.4527		.68835
43„34	.38012	.8463	.95118	1.3801	2.227	2.503	1.124	1.4510	1.631	.68920
43„40	.38242		.95451	1.3824				1.4483		.69046
43„45	.38434		.95730	1.3843				1.4461		.69151
43„50	.38628		.96008	1.3863				1.4439		.69256
43„55	.38822		.96288	1.3882				1.4417		.69361
44„ 0	.39016	.8566	.96569	1.3902	2.196	2.476	1.127	1.4396	1.623	.69466
44„ 5	.39212		.96850	1.3921				1.4374		.69570
44„10	.39408		.97133	1.3941				1.4352		.69675

φ deg.min.	x = sec. of φ−1	y	z =tan. of φ	t = sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of φ.
44„15	.39606		.97416	1.3960				1.4331		.69779
44„20	.39804		.97700	1.3980				1.4310		.69883
44„25	.40003	.8670	.97984	1.4000	2.168	2.449	1.130	1.4288	1.615	.69987
44„30	.40203		.98270	1.4020				1.4267		.70091
44„35	.40404		.98556	1.4040				1.4246		.70195
44„40	.40606		.98843	1.4060				1.4225		.70298
44„45	.40808		.99131	1.4081				1.4204		.70401
44„50	.41012	.8772	.99420	1.4101	2.140	2.424	1.133	1.4183	1.607	.70505
44„55	.41216		.99709	1.4122				1.4163		.70608
45„ 0	.41421		1.0000	1.4142				1.4142		.70711
45„ 5	.41627		1.0029	1.4163				1.4121		.70813
45„10	.41834		1.0058	1.4183				1.4101		.70916
45„14	.42001	.8871	1.0082	1.4200	2.112	2.400	1.136	1.4084	1.601	.70998
45„20	.42251		1.0017	1.4225				1.4061		.71121
45„25	.42461		1.0146	1.4246				1.4040		.71223
45„30	.42672		1.0176	1.4267				1.4020		.71325
45„35	.42883		1.0206	1.4288				1.4000		.71427
45„38	.43011	.8970	1.0223	1.4301	2.086	2.377	1.140	1.3988	1.594	.71488
45„40	.43100		1.0235	1.4310				1.3980		.71529
45„45	.43309		1.0265	1.4331				1.3961		.71630
45„50	.43524		1.0295	1.4352				1.3941		.71732
45„55	.43739		1.0325	1.4374				1.3921		.71833
46„ 1	.43999	.9066	1.0361	1.4400	2.060	2.355	1.143	1.3900	1.588	.71954
46„ 5	.44173		1.0385	1.4417				1.3882		.72035
46„10	.44391		1.0416	1.4439				1.3868		.72136
46„15	.44610		1.0446	1.4461				1.3843		.72236
46„20	.44831		1.0477	1.4483				1.3824		.72337
46„24	.45007	.9163	1.0501	1.4501	2.036	2.333	1.146	1.3809	1.583	.72417
46„30	.45274		1.0538	1.4527				1.3786		.72537
46„35	.45497		1.0568	1.4550				1.3767		.72637
46„40	.45721		1.0599	1.4572				1.3748		.72737
46„46	.45992	.9257	1.0637	1.4599	2.012	2.312	1.149	1.3725	1.577	.72857
46„50	.46173		1.0661	1.4617				1.3710		.72937
46„55	.46400		1.0692	1.4640				1.3692		.73036
47„ 0	.46628		1.0724	1.4663				1.3673		.73135
47„ 5	.46857		1.0755	1.4686				1.3655		.73234
47„ 8	.46995	.9350	1.0774	1.4699	1.989	2.292	1.152	1.3644	1.572	.73294
47„10	.47087		1.0786	1.4709				1.3636		.73333
47„15	.47319		1.0818	1.4732				1.3618		.73432
47„20	.47551		1.0849	1.4755				1.3600		.73531
47„25	.47784		1.0881	1.4778				1.3581		.73629
47„30	.48019	.9442	1.0913	1.4802	1.967	2.273	1.156	1.3563	1.567	.73728
47„35	.48254		1.0945	1.4825				1.3545		.73826
47„40	.48491		1.0977	1.4849				1.3527		.73924
47„45	.48728		1.1009	1.4873				1.3509		.74022
47„51	.49015	.9533	1.1048	1.4901	1.946	2.254	1.159	1.3488	1.563	.74139
47„55	.49207		1.1074	1.4921				1.3474		.74217
48„ 0	.49448		1.1106	1.4945				1.3456		.74314
48„ 5	.49690		1.1139	1.4969				1.3439		.74412
48„11	.49981	.9623	1.1178	1.4998	1.925	2.236	1.162	1.3418	1.559	.74528
48„15	.50177		1.1204	1.5018				1.3404		.74606
48„20	.50422		1.1237	1.5042				1.3386		.74702
48„25	.50668		1.1270	1.5067				1.3369		.74799
48„30	.50916		1.1303	1.5092				1.3352		.74896

φ	x = sec. of φ−1	y	z = tan. of φ	t = sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{x}{y}$	$\frac{t}{z}$ = cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of φ
deg.min.										
48″32	.51015	.9713	1.1316	1.5101	1.905	2.218	1.165	1.3345	1.555	.74934
48″35	.51165		1.1336	1.5116				1.3335		.74992
48″40	.51414		1.1369	1.5141				1.3318		.75088
48″45	.51665		1.1403	1.5166				1.3301		.75184
48″50	.51918		1.1436	1.5192				1.3284		.75280
48″52	.52019	.9800	1.1450	1.5202	1.885	2.201	1.168	1.3277	1.551	.75318
48″55	.52171		1.1470	1.5217				1.3267		.75375
49″ 0	.52425		1.1504	1.5242				1.3250		.75471
49″ 5	.52681		1.1537	1.5268				1.3233		.75566
49″11	52989	.9887	1.1578	1.5299	1.865	2.185	1.171	1.3213	1.547	.75680
49″15	.53196		1.1605	1.5320				1.3200		.75756
49″20	.53455		1.1640	1.5345				1.3184		.75851
49″25	.53715		1.1674	1.5371				1.3167		.75946
49″31	.54029	.9973	1.1715	1.5408	1.847	2.169	1.174	1.3148	1.544	.76059
49″35	.54240		1.1743	1.5424				1.3134		.76135
49″40	.54504		1.1778	1.5450				1.3118		.76229
49″45	54769		1.1812	1.5477				1.3102		.76323
49″49	.54982	1.0058	1.1840	1.5490	1.829	2.153	1.177	1.3089	1.541	.76398
49″55	.55303		1.1882	1.5530				1.3070		.76511
50″ 0	.55572		1.1917	1.5557				1.3054		.76604
50″ 5	.55843		1.1953	1.5584				1.3038		.76698
50″ 8	.56005	1.0142	1.1974	1.5600	1.811	2.138	1.181	1.3029	1.538	.76754
50″10	.56114		1.1988	1.5611				1.3022		.76791
50″15	.56387		1.2024	1.5639				1.3007		.76884
50″20	.56661		1.2059	1.5666				1.2991		.76977
50″26	.56992	1.0225	1.2102	1.5699	1.794	2.123	1.184	1.2972	1.535	.77088
50″30	.57213		1.2131	1.5721				1.2960		.77162
50″35	.57491		1.2167	1.5749				1.2944		.77255
50″40	.57771		1.2203	1.5777				1.2929		.77347
50″44	.57995	1.0307	1.2233	1.5799	1.777	2.109	1.187	1.2916	1.532	.77421
50″50	.58333		1.2276	1.5833				1.2898		.77531
50″55	.58617		1.2312	1.5862				1.2883		.77623
51″ 0	.58902		1.2349	1.5890				1.2868		.77715
51″ 2	.59016	1.038	1.2364	1.5902	1.761	2.095	1.190	1.2861	1.530	.77751
51″ 5	.59188		1.2386	1.5919				1.2852		.77806
51″10	.59475		1.2423	1.5947				1.2837		.77897
51″15	.59764		1.2460	1.5976				1.2822		.77988
51″19	.59996	1.0469	1.2490	1.6000	1.745	2.082	1.193	1.2810	1.528	.78061
51″25	.60346		1.2534	1.6034				1.2793		.78 70
51″30	.60639		1.2572	1.6064				1.2778		.78261
51″36	.60992	1.0549	1.2617	1.6099	1.729	2.069	1.196	1.2760	1.526	.78369
51″40	.61229		1.2647	1.6123				1.2748		.78442
51″45	.61526		1.2685	1.6153				1.2734		.78532
51″50	.61825		1.2723	1.6182				1.2719		.78622
51″53	.62005	1.0627	1.2746	1.6200	1.714	2.056	1.199	1.2710	1.525	.78675
51″55	.62125		1.2761	1.6212				1.2705		.78711
52″ 0	.62427		1.2799	1.6243				1.2690		.78801
52″ 5	.62730		1.2838	1.6273				1.2676		.78890
52″ 9	.62973	1.0706	1.2869	1.6297	1.699	2.043	1.202	1.2664	1.523	.78962
52″15	.63341		1.2915	1.6334				1.2647		.79069
52″20	.63648		1.2954	1.6365				1.2633		.79158
52″26	.64019	1.0782	1.3001	1.6402	1.685	2.031	1.206	1.2616	1.521	.79264
52″30	.64268		1.3032	1.6427				1.2605		.79335
52″35	.64580		1.3072	1.6458				1.2591		.79424

ϕ	x =sec. ofϕ−1	y	z =tan. of ϕ	t =sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ =cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ =sine of ϕ
deg. min										
52„40	.64894		1.3111	1.6489				1.2577		.79512
52„42	.65020	1.0859	1.3127	1.6502	1.671	2.019	1.209	1.2571	1.519	.79547
52„45	.65209		1.3151	1.6521				1.2563		.79600
52„50	.65526		1.3190	1.6553				1.2549		.79688
52„55	.65844		1.3230	1.6584				1.2535		.79776
52„57	.65972	1.0935	1.3246	1.6597	1.657	2.007	1.212	1.2530	1.518	.79811
53„ 0	.66164		1.3270	1.6616				1.2521		.79863
53„ 5	.66485		1.3311	1.6648				1.2508		.79951
53„10	.66809		1.3351	1.6681				1.2494		.80038
53„13	.67003	1.1009	1.3375	1.6700	1.643	1.996	1.215	1.2486	1.517	.80090
53„15	.67133		1.3392	1.6713				1.2480		.80125
53„20	.67460		1.3432	1.6746				1.2467		.80212
53„25	.67788		1.3473	1.6779				1.2453		.80299
53„28	.67985	1.1085	1.3498	1.6800	1.630	1.985	1.218	1.2445	1.516	.80351
53„30	.68117		1.3514	1.6812				1.2440		.80386
53„35	.68447		1.3555	1.6845				1.2427		.80472
53„40	.68781		1.3597	1.6878				1.2413		.80558
53„43	.68982	1.1159	1.3622	1.6898	1.617	1.974	1.221	1.2405	1.515	.80610
53„45	.69116		1.3638	1.6912				1.2400		.80644
53„50	.69452		1.3680	1.6945				1.2387		.80730
53„55	.69790		1.3722	1.6979				1.2374		.80816
53„58	.69994	1.1232	1.3747	1.6999	1.605	1.964	1.224	1.2366	1.514	.80867
54„ 0	.70130		1.3764	1.7013				1.2361		.80902
54„ 5	.70472		1.3806	1.7047				1.2348		.80987
54„10	.70815		1.3848	1.7081				1.2335		.81072
54„13	.71021	1.1304	1.3874	1.7102	1.592	1.954	1.227	1.2327	1.513	.81123
54„15	.71160		1.3891	1.7116				1.2322		.81157
54„20	.71506		1.3934	1.7151				1.2309		.81242
54„25	.71855		1.3976	1.7185				1.2296		.81327
54„27	.71995	1.1375	1.3994	1.7199	1.580	1.944	1.230	1.2291	1.512	.81361
54„30	.72205		1.4019	1.7220				1.2283		.81411
54„35	.72557		1.4063	1.7256				1.2270		.81496
54„41	.72982	1.1446	1.4114	1.7298	1.568	1.934	1.233	1.2255	1.5115	.81600
54„45	.73267		1.4150	1.7327				1.2245		.81664
54„50	.73624		1.4193	1.7362				1.2233		.81748
54„55	.73983	1.1517	1.4237	1.7398	1.556	1.924	1.236	1.2220	1.5108	.81832
55„ 0	.74345		1.4281	1.7434				1.2207		.81915
55„ 5	.74708		1.4326	1.7471				1.2195		.81998
55„ 9	.75000	1.1587	1.4361	1.7500	1.545	1.915	1.239	1.2185	1.5103	.82065
55„13	.75440		1.4415	1.7544				1.2171		.82165
55„20	.75808		1.4460	1.7581				1.2158		.82247
55„23	.76031	1.1657	1.4487	1.7603	1.534	1.906	1.242	1.2151	1.5098	.82297
55„25	.76179		1.4505	1.7618				1.2146		.82330
55„30	.76552		1.4550	1.7655				1.2134		.82413
55„36	.77001	1.1726	1.4605	1.7700	1.523	1.897	1.246	1.2119	1.5094	.82511
55„40	.77303		1.4641	1.7730				1.2110		.82577
55„45	.77681		1.4687	1.7768				1.2098		.82659
55„49	.77986	1.1795	1.4724	1.7800	1.512	1.888	1.249	1.2088	1.5092	.82734
55„55	.78444		1.4779	1.7844				1.2074		.82822
56„ 0	.78829		1.4826	1.7883				1.2062		.82904
56„ 2	.78984	1.1861	1.4842	1.7898	1.501	1.879	1.252	1.2057	1.5091	.82936
56„ 5	.79216		1.4872	1.7922				1.2050		.82985
56„10	.79604		1.4919	1.7960				1.2039		.83066
56„15	.79995	1.1929	1.4967	1.7999	1.491	1.871	1.255	1.2027	1.5089	.83147

φ	x =sec. of $\varphi-1$	y	z = tan. of φ	t = sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ =cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of φ
deg. min.										
56„20	.80388		1.5013	1.8039				1.2015		.83228
56„25	.80783		1.5061	1.8078				1.2004		.83308
56„28	.81021	1.1995	1.5089	1.8102	1.481	1.863	1.258	1.1997	1.5089	.83356
56„30	.81180		1.5108	1.8118				1.1992		.83389
56„35	.81579		1.5156	1.8158				1.1980		.83469
56„40	.81981	1.2061	1.5204	1.8198	1.471	1.855	1.261	1.1969	1.5090	.83549
56„45	.82384		1.5252	1.8238				1.1958		.83629
56„50	.82790		1.5301	1.8279				1.1946		.83708
56„53	.83034	1.2127	1.5330	1.8303	1.461	1.847	1.264	1.1939	1.5090	.83756
57„ 0	.83608		1.5399	1.8361				1.1924		.83867
57„ 5	.84020	1.2191	1.5448	1.8402	1.451	1.839	1.267	1.1912	1.5093	.83946
57„10	.84435		1.5497	1.8443				1.1901		.84025
57„15	.84852		1.5547	1.8485				1.1890		.84104
57„17	.85019	1.2256	1.5567	1.8502	1.442	1.831	1.270	1.1886	1.5095	.84135
57„20	.85271		1.5596	1.8527				1.1879		.84182
57„25	.85692		1.5646	1.8569				1.1868		.84261
57„29	.86031	1.2321	1.5687	1.8603	1.433	1.824	1.273	1.1859	1.5097	.84323
57„35	.86542		1.5747	1.8654				1.1846		.84417
57„40	.86990	1.2384	1.5798	1.8699	1.423	1.816	1.276	1·1835	1.5100	.84495
57„45	.87401		1.5849	1.8740				1.1824		.84573
57„50	.87834		1.5900	1.8783				1.1813		.84650
57„52	.88008	1.2447	1.5921	1.8801	1.414	1.809	1.279	1.1809	1.5104	.84681
57„55	.88270		1.5952	1.8827				1.1802		.84728
58„ 0	.88708		1.6003	1.8871				1.1792		.84805
58„ 3	.88972	1.2510	1.6034	1.8897	1.416	1.802	1.282	1.1785	1.5108	.84851
58„ 5	.89148		1.6055	1.8915				1.1781		.84882
58„10	.89591		1.6107	1.8959				1.1770		.84959
58„15	.90037	1.2572	1.6160	1.9004	1.397	1.795	1.285	1.1760	1.5114	.85035
58„20	.90485		1.6212	1.9048				1.1749		.85112
58„26	.91025	1.2633	1.6276	1.9102	1.388	1.788	1.288	1.1737	1.5119	.85203
58„30	.91388		1.6318	1.9139				1.1728		.85264
58„35	.91844		1.6372	1.9184				1.1718		.85340
58„37	.92026	1.2694	1.6393	1.9203	1.380	1.781	1.291	1.1714	1.5126	.85370
58„40	.92302		1.6426	1.9230				1.1707		.85416
58„45	.92762		1.6479	1.9276				1.1697		.85491
58„48	.93040	1.2755	1.6512	1.9304	1.372	1.775	1.294	1.1691	1.5132	.85536
58„50	.93226		1.6534	1.9323				1.1687		.85566
58„55	.93692		1.6588	1.9369				1.1676		.85642
58„58	.93973	1.2816	1.6621	1.9397	1.363	1.769	1.297	1.1670	1.5137	.85687
59„ 0	.94160		1.6643	1.9416				1.1666		.85717
59„ 5	.94632		1.6698	1.9463				1.1656		.85791
59„ 9	.95011	1.2876	1.6742	1.9501	1.355	1.762	1.300	1.1648	1.5144	.85851
59„15	.95582		1.6808	1.9558				1.1636		.85941
59„19	.95966	1.2935	1.6853	1.9600	1.347	1.756	1.303	1.1628	1.5153	.86000
59„25	.96544		1.6920	1.9654				1.1616		.86089
59„30	.97029	1.2994	1.6977	1.9703	1.340	1.750	1.306	1.1606	1.5162	.86163
59„35	.97517		1.7033	1.9752				1.1596		.86237
59„40	.98008	1.3053	1.7090	1.9801	1.332	1.744	1.309	1.1586	1.5174	.86310
50„45	.98502		1.7147	1.9850				1.1576		.86383
59„50	.98998	1.3111	1.7205	1.9900	1.324	1.738	1.312	1.1566	1.5178	.86457
59„55	.99498		1.7262	1.9950				1.1557		.86530
60„ 0	1.0000	1.3169	1.7321	2.0000	1.317	1.732	1.315	1.1547	1.5186	.86602
60„ 5	1.0050		1.7379	2.0050				1.1537		.86675
60„10	1.0101		1.7437	2.0101				1.1528		.86748

ϕ	x = sec. of $\phi-1$	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
deg. min										
60„15	1.0152		1.7496	2.0152				1.1518		.86820
60„20	1.0204	1.3284	1.7556	2.0204	1.302	1.721	1.321	1.1508	1.5206	.86892
60„25	1.0255		1.7615	2.0256				1.1499		.86964
60„30	1.0307		1.7675	2.0307				1.1489		.87036
60„35	1.0360		1.7735	2.0360				1.1480		.87107
60„39	1.0402	1.3397	1.7783	2.0402	1.288	1.710	1.327	1.1473	1.5228	.8[illegible]164
60„45	1.0465		1.7856	2.0466				1.1461		.87250
60„50	1.0519		1.7917	2.0519				1.1452		.87321
60„55	1.0572		1.7979	2.0573				1.1443		.87391
60„58	1.0615	1.3509	1.8015	2.0615	1.274	1.699	1.333	1.1437	1.5249	.87434
61„ 0	1.0627		1.8040	2.0627				1.1433		.87462
61„ 5	1.0681		1.8102	2.0680				1.1424		.87532
61„10	1.0736		1.8165	2.0735				1.1415		.87603
61„16	1.0801	1.3619	1.8240	2.0801	1.261	1.689	1.339	1.1404	1.5278	.87687
61„20	1.0846		1.8291	2.0846				1.1397		.87743
61„25	1.0901		1.8354	2.0901				1.1388		.87812
61„30	1.0957		1.8418	2.0957				1.1379		.87882
61„34	1.1002	1.3728	1.8469	2.1002	1.248	1.679	1.345	1.1372	1.5297	.87937
61„40	1.1070		1.8546	2.1070				1.1361		.88020
61„45	1.1127		1.8611	2.1127				1.1352		.88089
61„51	1.1196	1.3836	1.8689	2.1196	1.235	1.669	1.351	1.1341	1.5322	.88171
61„55	1.1242		1.8741	2.1242				1.1334		.88226
62„ 0	1.1300		1.8807	2.1300				1.1326		.88295
62„ 5	1.1359		1.8873	2.1359				1.1317		.88363
62„ 8	1.1394	1.3942	1.8913	2.1394	1.223	1.660	1.357	1.1312	1.5349	.88404
62„10	1.1418		1.8940	2.1418				1.1308		.88431
62„15	1.1477		1.9007	2.1477				1.1300		.88499
62„20	1.1536		1.9074	2.1536				1.1291		.88566
62„25	1.1597	1.4047	1.9142	2.1597	1.211	1.651	1.363	1.1282	1.5377	.88634
62„30	1.1657		1.9210	2.1657				1.1274		.88701
62„35	1.1717		1.9278	2.1717				1.1265		.88768
62„40	1.1779		1.9347	2·1779				1.1257		.88835
62„42	1.1803	1.4151	1.9374	2.1803	1.199	1.642	1.369	1.1253	1.5405	.88862
62„45	1.1840		1.9416	2.1840				1.1248		.88902
62„50	1.1902		1.9486	2.1902				1.1240		.88968
62„55	1.1964		1.9556	2.1964				1.1232		.89034
62„58	1.2002	1.4253	1.9598	2.2002	1.188	1.633	1.375	1.1227	1.5435	.89074
63„ 0	1.2027		1.9626	2.2027				1.1223		.89101
63„ 5	1.2090		1.9697	2.2090				1.1215		.89167
63„10	1.2153		1.9768	2.2153				1.1207		.89232
63„15	1.2217		1.9840	2.2217				1.1198		.89298
63„21	1.2295	1.4406	1.9926	2.2295	1.171	1.620	1.384	1.1189	1.5480	.89376
63„25	1.2346		1.9984	2.2346				1.1182		.89428
63„30	1.2412		2.0057	2.2412				1.1174		.89493
63„35	1.2477		2.0130	2.2477				1.1166		.89558
63„40	1.2543		2.0204	2.2543				1.1158		.89623
63„44	1.2596	1.4555	2.0263	2.2596	1.155	1.608	1.392	1.1151	1.5527	.89674
63„50	1.2677		2.0352	2.2677				1.1142		.89751
63„55	1.2744		2.0428	2.2744				1.1134		.89815
64„ 0	1.2812		2.0503	2.2812				1.1126		.89879
64„ 6	1.2894	1.4702	2.0594	2.2894	1.140	1.597	1.401	1.1117	1.5576	.89956
64„10	1.2949		2.0655	2.2949				1.1110		.90006
64„15	1.3018		2.0732	2.3018				1.1102		.90070
64„20	1.3087		2.0809	2.3087				1.1095		.90133

ϕ	x = sec of ϕ	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{t}{z}$ = sine of ϕ
deg. min.										
64„25	1.3158		2.0887	2.3158				1.1087		.90196
64„28	1.3200	1.4846	2.0934	2.3200	1.125	1.586	1.410	1.1082	1.5627	.90233
64„30	1.3228		2.0965	2.3228				1.1079		.90258
64„35	1.3299		2.1044	2.3299				1.1072		.90321
64„40	1.3371		2.1123	2.3371				1.1064		.90383
64„45	1.3443		2.1203	2.3443				1.1056		.90445
64„49	1.3501	1.4988	2.1267	2.3501	1.110	1.575	1.419	1.1050	1.5679	.90495
64„55	1.3588		2.1364	2.3588				1.1041		.90569
65„ 0	1.3662		2.1445	2.3662				1.1034		.90631
65„ 5	1.3736		2.1527	2.3736				1.1026		.90692
65„ 9	1.3800	1.5128	2.1592	2.3800	1.096	1.565	1.428	1.1020	1.5732	.90741
65„15	1.3886		2.1692	2.3886				1.1011		.90814
65„20	1.3961		2.1775	2.3961				1.1004		.90875
65„25	1.4037		2.1859	2.4037				1.0997		.90936
65„29	1.4099	1.5266	2.1926	2.4099	1.083	1.555	1.436	1.0991	1.5787	.90984
65„35	1.4191		2.2028	2.4191				1.0982		.91056
65„40	1.4269		2.2113	1.4269				1.0975		.91116
65„45	1.4347		2.2199	2.4347				1.0968		.91176
65„50	1.4426		2.2286	2.4426				1.0961		.91236
65„55	1.4506	1.5446	2.2373	2.4506	1.065	1.542	1.448	1.0953	1.5862	.91295
66„ 0	1.4586		2.2460	2.4586				1.0946		.91354
66„ 5	1.4666		2.2549	2.4666				1.0939		.91414
66„10	1.4748		2.2637	2.4748				1.0932		.91472
66„15	1.4829		2.2727	2.4829				1.0925		.91531
66„19	1.4895	1.5624	2.2799	2.4895	1.049	1.530	1.460	1.0920	1.5937	.91578
66„25	1.4995		2.2799	2.4995				1.0911		.91648
66„30	1.5078		2.2998	2.5078				1.0904		.91706
66„35	1.5163		2.3090	2.5163				1.0897		.91764
66„40	1.5247		2.3183	2.5247				1.0891		.91822
66„43	1.5299	1.5798	2.3238	2.5299	1.033	1.519	1.471	1.0887	1.6015	.91856
66„45	1.5333		2.3276	2.5333				1.0884		.91879
66„50	1.5419		2.3369	2.5419				1.0877		.91936
66„55	1.5506		2.3464	2.5506				1.0870		.91993
67„ 0	1.5593		2.3558	2.5593				1.0864		.92050
67„ 6	1.5700	1.5969	2.3673	2.5700	1.017	1.508	1.483	1.0856	1.6094	.92118
67„10	1.5770		2.3750	2.5770				1.0850		.92164
67„15	1.5859		2.3847	2.5859				1.0844		.92220
67„20	1.5949		2.3945	2.5949				1.0837		.92276
67„25	1.6040		2.4043	2.6040				1.0830		.92332
67„28	1.6095	1.6135	2.4102	2.6095	1.002	1.497	1.494	1.0826	1.6176	.92366
67„30	1.6131		2.4142	2.6131				1.0824		.92388
67„35	1.6223		2.4242	2.6223				1.0817		.92443
67„40	1.6316		2 4342	2.6316				1.0811		.92499
67„45	1.6410		2.4443	2.6410				1.0804		.92554
67„50	1.6504		2 4545	2.6504				1.0798		.92609
67„55	1.6600	1.6341	2.4648	2.6600	.9844	1.485	1.508	1.0792	1.6278	.92664
68„ 0	1.6695		2.4751	2.6695				1.0785		.92718
68„ 5	1.6791		2.4855	2.6791				1.0779		.92773
68„10	1.6888		2.4960	2.6888				1.0773		.92827
68„15	1.6986		2.5065	2.6986				1.0766		.92881
68„21	1.7105	1.6541	2.5192	2.7105	.9673	1.473	1.523	1.0759	1.6384	.92945
68„25	1.7185		2.5279	2.7185				1.0754		.92988
68„30	1.7285		2.5386	2.7285				1.0748		.93042
68„35	1.7386		2.5495	2.7386				1.0742		.93095

φ	x = sec. of φ−1	y	z = tan. of φ	t = sec. of φ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of φ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of φ
deg. min.										
68„40	1.7488		2.5605	2.7488				1.0736		.93148
68„45	1.7591	1.6738	2.5715	2.7591	.9510	1.462	1.537	1.0730	1.6489	.93201
68„50	1.7694		2.5826	2.7694				1.0723		.93253
68„55	1.7799		2.5938	2.7799				1.0717		.93306
69	1.7904		2.6050	2.7904				1.0711		.93358
69„5	1.8010		2.6165	2.8010				1.0705		.93410
69„9	1.8096	1.6930	2.6256	2.8096	.9354	1.451	1.551	1.0701	1.6596	.93451
69„15	1.8225		2.6394	2.8225				1.0694		.93513
69„20	1.8334		2.6511	2.8334				1.0688		.93565
69„25	1.8444		2.6628	2.8444				1.0682		.93616
69„30	1.8554		2.6746	2.8554				1.0676		.93667
69„32	1.8600	1.7118	2.6794	2.8600	.9203	1.441	1.565	1.0674	1.6707	.93688
69„35	1.8666		2.6865	2.8666				1.0670		.93718
69„40	1.8778		2.6985	2.8778				1.0664		.93769
69„45	1.8892		2.7106	2.8892				1.0659		.93819
69„50	1.9006		2.7228	2.9006				1.0653		.93869
69„54	1.9098	1.7303	2.7326	2.9098	.9059	1.431	1.579	1.0649	1.6818	.93909
70	1.9238		2.7475	2.9238				1.0642		.93969
70„ 5	1.9355		2.7600	2.9355				1.0636		.94019
70„10	1.9474		2.7725	2.9474				1.0631		.94068
70„15	1.9593	1.7484	2.7852	2.9593	.8920	1.421	1.593	1.0625	1.6930	.94118
70„20	1.9713		2.7980	2.9713				1.0619		.94167
70„25	1.9835		2.8109	2.9835				1.0614		.94215
70„30	1.9957		2.8239	2.9957				1.0608		.94264
70„36	2.0106	1.7662	2.8396	3.0106	.8787	1.412	1.607	1.0602	1.7042	.94322
70„40	2.0206		2.8502	3.0206				1.0598		.94361
70„45	2.0331		2.8636	3.0331				1.0592		.94409
70„50	2.0458		2.8770	3.0458				1.0587		.94457
70„56	2.0612	1.7836	2.8933	3.0612	.8658	1.404	1.621	1.0580	1.7156	.94514
71„	2.0715		2.9042	3.0715				1.0576		.94552
71„ 5	2.0846		2.9180	3.0846				1.0571		.94599
71„10	2.0977		2.9319	3.0977				1.0566		.94646
71„15	2.1110	1.8008	2.9459	3.1110	.8535	1.396	1.635	1.0560	1.7270	.94693
71„20	2.1244		2.9600	3.1244				1.0555		.94740
71„25	2.1379		2.9743	3.1379				1.0550		.94786
71„30	2.1515		2.9887	3.1515				1.0545		.94832
71„33	2.1600	1.8176	2.9974	3.1600	.8415	1.388	1.649	1.0542	1.7387	.94860
71„35	2.1653		3.0031	3.1653				1.0540		.94878
71„40	2.1792		3.0178	3.1792				1.0535		.94924
71„45	2.1932		3.0326	3.1932				1.0530		.94970
71„51	2.2102	1.8342	3.0505	3.2102	.8300	1.380	1.663	1.0524	1.7501	.95024
71„55	2.2216		3.0625	3.2216				1.0520		.95061
72„	2.2361		3.0777	3.2361				1.0515		.95106
72„ 5	2.2506		3.0930	3.2506				1.0510		.95150
72„ 8	2.2594	1.8505	3.1022	3.2594	.8188	1.373	1.677	1.0507	1.7617	.95177
72„10	2.2653		3.1084	3.2653				1.0505		.95195
72„15	2.2801		3.1240	3.2801				1.0500		.95240
72„20	2.2951		3.1397	3.2951				1.0495		.95284
72„25	2.3102		3.1556	3.3102				1.0490		.95328
72„28	2.3194	1.8696	3.1652	3.3194	.8059	1.365	1.693	1.0487	1.7758	.95354
72„30	2.3255		3.1716	3.3255				1.0485		.95372
72„35	2.3409		3.1877	3.3409				1.0480		.95415
72„40	2.3565		3.2041	3.3565				1.0476		.95459
72„45	2.3722		3.2205	3.3722				1.0471		.95502

ϕ	x = sec. of $\phi-1$	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
deg. min.										
72″47	2.3785	1.8888	3.2271	3.3785	.7934	1.357	1.710	1.0469	1.7900	.95519
72″50	2.3881		3.2371	3.3881				1.0466		.95545
72″55	2.4041		3.2539	3.4041				1.0462		.95588
73″	2.4203		3.2708	3.4203				1.0457		.95630
73″ 6	2.4400	1.9067	3.2914	3.4400	.7814	1.349	1.726	1.0451	1.8042	.95681
73″10	2.4532		3.3052	3.4532				1.0448		.95715
73″15	2.4699		3.3226	3.4699				1.0443		.95757
73″20	2.4867		3.3402	3.4867				1.0438		.95800
73″24	2.5003	1.9248	3.3544	3.5003	.7699	1.342	1.743	1.0435	1.8184	.95832
73″30	2.5209		3.3759	3.5209				1.0429		.95882
73″35	2.5383		3.3941	3.5383				1.0425		.95923
73″41	2.5594	1.9426	3.4160	3.5594	.7588	1.335	1.759	1.0420	1.8326	.95972
73″45	2.5736		3.4308	3.5736				1.0416		.96005
73″50	2.5915		3.4495	3.5915				1.0412		.96046
73″55	2.6096		3.4684	2.6096				1.0407		.96086
73″58	2.6206	1.9600	3.4798	3.6206	.7481	1.328	1.775	1.0404	1.847	.96110
74″	2.6279		3.4874	3.6279				1.0403		.96126
74″ 5	2.6464		3.5066	3.6464				1.0399		.96166
74″10	2.6651		3.5261	3.6651				1.0394		.96206
74″14	2.6802	1.9770	3.5418	3.6802	.7377	1.321	1.791	1.0391	1.861	.96238
74″20	2.7031		3.5656	3.7031				1.0386		.96285
74″25	2.7224		3.5856	3.7224				1.0382		96324
74″29	2.7381	1.9938	3.6018	3.7381	.7277	1.315	1.808	1.0378	1.876	.96355
74″35	2.7617		3.6263	3.7617				1.0373		.96402
74″40	2.7817		3.6470	3.7817				1.0369		.96440
74″45	2.8018	2.0103	3.6680	3.8018	.7180	1.309	1.824	1.0365	1.890	.96479
74″50	2.8222		3.6891	3.8222				1.0361		.96517
74″55	2.8428		3.7105	3.8428				1.0357		.96555
74″59	2.8595	2.0266	3.7277	3.8595	.7086	1.304	1.840	1.0354	1.905	.96585
75″ 5	2.8848		3.7539	3.8848				1.0349		.96630
75″10	2.9061		3.7759	3.9061				1.0345		.96667
75″13	2.9190	2.0425	3.7893	3.9190	.6995	1.298	1.856	1.0342	1.919	.96690
75″15	2.9277		3.7983	3.9277				1.0341		.96704
75″20	2.9495		3.8208	3.9495				1.0337		.96741
75″25	2.9716		3.8436	3.9716				1.0333		.96778
75″27	2.9805	2.0582	3.8528	3.9805	.6907	1.293	1.872	1.0331	1.934	.96793
75″30	2.9939		3.8667	3.9939				1.0329		.96815
75″35	3.0165		3.8900	4.0165				1.0325		.96851
75″40	3.0394	2.0737	3.9136	4.0394	.6821	1.288	1.888	1.0321	1.948	.96887
75″45	3.0625		3.9375	4.0625				1.0317		.96923
75″50	3.0859		3.9616	4.0859				1.0314		.96959
75″53	3.1001	2.0889	3.9763	4.1001	.6738	1.283	1.903	1.0311	1.962	.96980
75″55	3.1096		3.9861	4.1096				1.0310		.96994
76″	3.1336		4.0108	4.1336				1.0306		.97030
76″ 5	3.1578		4.0358	4.1578				1.0302		.97065
76″ 7	3.1676	2.1063	4.0459	4.1676	.6644	1.277	1.922	1.0301	1.980	.97079
76″10	3.1824		4.0611	4.1824				1.0299		.97099
76″15	3.2072		4.0867	4.2072				1.0295		.97134
76″21	3.2375	2.1234	4.1178	4.2375	.6554	1.272	1.940	1.0291	1.997	.97175
76″25	3.2579		4.1388	4.2579				1.0288		.97203
76″30	3.2837		4.1653	4.2837				1.0284		.97237
76″35	3.3100	2.1403	4.1921	4.3100	.6466	1.267	1.959	1.0281	2.014	.97271
76″40	3.3362		4.2193	4.3362				1.0277		.97304
76″45	3.3630		4.2468	4.3630				1.0273		.97338

φ	x = sec. of φ−1	y	z = tan. of φ	t = sec. of φ	y/x	z/x	z/y	t/z = cosec. of	t/y	z/t = sine of φ
deg. min.										
76″48	3.3792	2.1569	4.2635	4.3792	.6381	1.262	1.977	1.0271	2.031	.97358
76″50	3.3901		4.2747	4.3901				1.0270		.97371
76″55	3.4176		4.3029	4.4176				1.0266		.97404
77″ 1	3.4510	2.1731	4.3372	4.4510	.6299	1.257	1.995	1.0262	2.048	.97443
77″ 5	2.4736		4.3604	4.4736				1.0260		.97470
77″10	2.5021		4.3897	4.5021				1.0256		.97502
77″13	3.5195	2.1891	4.4074	4.5195	.6219	1.252	2.014	1.0254	2.065	.97521
77″15	2.5311		4.4194	4.5311				1.0253		.97534
77″20	2.5604		4.4494	4.5604				1.0249		.97566
77″25	3.5901	2.2049	4.4799	4.5901	.6142	1.248	2.032	1.0246	2.082	.97598
77″30	3.6202		4.5107	4.6202				1.0243		.97630
77″36	3.6569	2.2203	4.5483	4.6569	.6066	1.244	2.050	1.0239	2.099	.97667
77″40	2.6817		4.5736	4.6817				1.0236		.97692
77″45	2.7130		4.6057	4.7130				1.0233		.97723
77″48	3.7320	2.2356	4.6252	4.7320	.5994	1.239	2.068	1.0231	2.116	.97742
77″50	2.7448		4.6382	4.7448				1.0230		.97754
77″55	2.7770		4.6712	4.7770				1.0227		.97784
77″59	3.8032	2.2507	4.6979	4.8032	.5923	1.235	2.086	1.0224	2.133	.97809
78″ 5	3.8429		4.7385	4.8429				1.0220		.97845
78″11	3.8833	2.2676	4.7798	4.8833	.5844	1.231	2.106	1.0216	2.152	.97881
78″15	3.9106		4.8077	4.9106				1.0214		.97904
78″20	3.9452		4.8430	4.9452				1.0211		.97934
78″22	3.9591	2.2842	4.8573	4.9591	.5768	1.227	2.127	1.0210	2.187	.97946
78″25	3.9802		4.8788	4.9802				1.0208		.97963
78″30	4.0158		4.9151	5.0158				1.0205		.97992
78″33	4.0375	2.3005	4.9372	5.0375	.5694	1.223	2.147	1.0203	2.191	.98010
78″35	4.0520		4.9520	5.0520				1.0202		.98021
78″40	4.0886		4.9894	5.0886				1.0199		.98050
78″44	4.1183	2.3166	5.0197	5.1183	.5623	1.219	2.168	1.0196	2.210	.98073
78″50	4.1636		5.0658	5.1636				1.0193		.98107
78″55	4.2019	2.3324	5.1049	5.2019	.5553	1.215	2.188	1.0190	2.230	.98135
79	4.2408		5.1445	5.2408				1.0187		.98163
79″ 5	4.2804	2.3479	5.1848	5.2804	.5486	1.211	2.208	1.0184	2.249	.98190
79″10	4.3205		5.2257	5.3205				1.0181		.98218
79″15	4.3612	2.3632	5.2671	5.3612	.5420	1.208	2.228	1.0179	2.268	.98245
79″20	4.4026		5.3093	5.4026				1.0176		.98272
79″26	4.4532	2.3802	5.3607	5.4532	.5349	1.204	2.251	1.0172	2.290	.98304
79″30	4.4874		5.3955	5.4874				1.0170		.98325
79″36	4.5396	2.3969	5.4486	5.5396	.5280	1.200	2.273	1.0167	2.311	.98357
79″40	4.5749		5.4845	5.5749				1.0165		.98378
79″46	4.6288	2.4132	5.5393	5.6288	.5212	1.197	2.296	1.0162	2.333	.98409
79″50	4.6653		5.5764	5.6653				1.0159		.98430
79″56	4.7210	2.4293	5.6329	5.7210	.5147	1.193	2.318	1.0156	2.355	.98460
80	4.7588		5.6713	5.7588				1.0154		.98481
80″ 5	4.8067	2.4452	5.7199	5.8067	.5084	1.190	2.341	1.0152	2.376	.98506
80″10	4.8554		5.7694	5.8554				1.0149		.98531
80″15	4.9049	2.4607	5.8197	5.9049	.5022	1.187	2.363	1.0146	2.398	.98555
80″20	4.9554		5.8708	5.9554				1.0144		.98581
80″24	4.9963	2.4778	5.9123	5.9963	.4956	1.183	2.388	1.0142	2.421	.98600
80″30	5.0588		5.9758	6.0588				1.0139		.98629
80″34	5.1013	2.4946	6.0188	6.1013	.4891	1.180	2.412	1.0137	2.445	.98648
80″40	5.1661		6.0844	6.1661				1.0134		.98676
80″43	5.1990	2.5111	6.1178	6.1990	.4829	1.177	2.437	1.0133	2.469	.98690
80″45	5.2211		6.1402	6.2211				1.0132		.98700

ϕ	x = sec. of $\phi - 1$	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
deg. min.										
80„50	5.2772		6.1970	6.2772				1.0129		.98723
80„52	5.2999	2.5273	6.2200	6.2999	.4768	1.174	2.461	1.0128	2.492	.98732
80„55	5.3348		6.2548	6.3348				1.0127		.98746
81„ 1	5.4042	2.5433	6.3257	6.4042	.4710	1.171	2.486	1.0124	2.516	.98773
81„ 5	5.4517		6.3737	6.4517				1.0122		.98791
81„ 9	5.4999	2.5589	6.4225	6.4999	.4653	1.168	2.510	1.0120	2.540	.98809
81„15	5.5736		6.4971	6.5736				1.0118		.98836
81„17	5.5986	2.5743	6.5223	6.5986	.4597	1.165	2.534	1.0116	2.564	.98845
81„20	5.6363		6.5605	6.6363				1.0115		.98858
81„25	5.7003	2.5895	6.6252	6.7003	.4543	1.163	2.558	1.0113	2.587	.98880
81„30	5.7655		6.6912	6.7655				1.0111		.98902
81„33	5.8052	2.6046	6.7313	6.8052	.4491	1.160	2.582	1.0110	2.611	.98914
81„35	5.8320		6.7584	6.8320				1.0109		.98923
81„40	5.8998	2.6195	6.8269	6.8998	.4440	1.157	2.606	1.0107	2.634	.98944
81„45	5.9690		6.8969	6.9690				1.0104		.98965
81„47	5.9971	2.6339	6.9252	6.9971	.4390	1.155	2.630	1.0104	2.657	.98973
81„50	6.0396		6.9682	7.0396				1.0102		.98986
81„54	6.0972	2.6482	7.0264	7.0972	.4341	1.152	2.654	1.0101	2.681	.99002
82„ 1	6.2002	2.6624	7.1304	7.2002	.4294	1.150	2.678	1.0098	2.704	.99031
82„ 5	6.2604		7.1912	7.2604				1.0096		.99047
82„ 8	6.3063	2.6763	7.2375	7.3063	.4248	1.148	2.702	1.0095	2.728	.99059
82„10	6.3372		7.2687	7.3372				1.0094		.99067
82„14	6.4000	2.6900	7.3319	7.4000	.4203	1.146	2.726	1.0092	2.751	.99083
82„20	6.4957	2.7035	7.4287	7.4957	.4159	1.144	2.749	1.0090	2.774	.99110
82„26	6.5942	2.7169	7.5281	7.5942	.4117	1.142	2.773	1.0088	2.797	.99130
82„30	6.6613		7.5957	7.6613				1.0086		.99144
82„32	6.6953	2.7301	7.6300	7.6953	.4075	1.140	2.797	1.0085	2.820	.99152
82„38	6.7992	2.7431	7.7348	7.7992	.4034	1.138	2.820	1.0083	2.843	.99174
82„40	6.8344		7.7703	7.8344				1.0082		.99182
82„45	6.9240		7.8606	7.9240				1.0081		.99200
82„49	6.9971	2.7686	7.9344	7.9971	.3955	1.134	2.867	1.0079	2.890	.99215
82„55	7.1095		8.0475	8.1095				1.0077		.99237
83„ 0	7.2055	2.7935	8.1443	8.2055	.3880	1.130	2.913	1.0075	2.935	.99254
83„ 5	7.3039		8.2434	8.3039				1.0073		.99272
83„10	7.4047	2.8177	8.3450	8.4047	.3808	1.127	2.960	1.0071	2.981	.99290
83„15	7.5079		8.4490	8.5079				1.0070		.99307
83„19	7.5924	2.8415	8.5340	8.5924	.3739	1.124	3.006	1.0068	3.027	.99320
83„25	7.7223		8.6648	8.7223				1.0066		.99341
83„29	7.8112	2.8646	8.7542	8.8112	.3673	1.121	3.052	1.0065	3.072	.99354
83„35	7.9479		8.8918	8.9479				1.0063		.99373
83„37	7.9944	2.8873	8.9387	8.9944	.3609	1.118	3.098	1.0062	3.117	.99380
83„40	8.0651		9.0098	9.0651				1.0061		.99390
83„46	8.2100	2.9094	9.1555	9.2100	.3548	1.115	3.143	1.0059	3.162	.99409
83„50	8.3092		9.2553	9.3092				1.0058		.99421
83„54	8.4105	2.9309	9.3572	9.4105	.3489	1.113	3.189	1.0057	3.207	.99434
84„ 0	9.5668		9.5144	9.5668				1.0055		.99452
84„ 1	8.5933	2.9521	9.5411	9.5933	.3433	1.110	3.234	1.0055	3.252	.99455
84„ 5	8.7010		9.6493	9.7010				1.0054		.99467
84„ 9	8.8112	2.9728	9.7601	9.8112	.3378	1.108	3.279	1.0052	3.297	.99479
84„16	9.0101	2.9932	9.9601	10.010	.3326	1.106	3.324	1.0050	3.341	.99500
84„22	9.1872	3.0134	10.1381	10.187	.3275	1.103	3.368	1.0049	3.385	.99517
84„29	9.4020	3.0326	10.3538	10.402	.3226	1.101	3.413	1.0047	3.429	.99537
84„35	9.5934	3.0517	10.5561	10.593	.3179	1.099	3.458	1.0045	3.473	.99553
84„41	9.7921	3.0705	10.7457	10.792	.3133	1.097	3.502	1.0043	3.517	.99570

ϕ deg.min.	x = sec. of $\phi-1$	y	z = tan. of ϕ	t = sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ = cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ = sine of ϕ
84″47	9.9984	3.0890	10.9528	10.998	.3089	1.095	3.546	1.0042	3.561	.99586
84″53	10.213	3.1070	11.1681	11.213	.3046	1.094	3.590	1.0040	3.605	.99601
84″58	10.398	3.1248	11.3540	11.398	.3005	1.092	3.634	1.0039	3.648	.99614
85″ 3	10.590	3.1422	11.5460	11.590	.2964	1.090	3.678	1.0037	3.692	.99627
85″ 8	10.787	3.1593	11.7448	11.787	.2925	1.089	3.722	1.0036	3.735	.99639
85″13	10.992	3.1762	11.9504	11.992	.2887	1.087	3.765	1.0035	3.778	.99652
85″18	11.204	3.1929	12.1632	12.204	.2851	1.086	3.808	1.0034	3.821	.99664
85″22	11.379	3.2092	12.3390	12.379	.2815	1.084	3.851	1.0033	3.864	.99673
85″27	11.606	3.2252	12.5660	12.606	.2780	1.083	3.894	1.0032	3.907	.99685
85″31	11.793	3.2410	12.7536	12.793	.2747	1.081	3.937	1.0031	3.949	.99694
85″35	11.985	3.2566	12.9469	12.985	.2714	1.080	3.980	1.0030	3.992	.99703
85″39	12.184	3.2719	13.1461	13.184	.2682	1.079	4.023	1.0029	4.034	.99712
85″43	12.389	3.2870	13.3515	13.389	.2651	1.078	4.065	1.0028	4.077	.99721
85″47	12.600	3.3019	13.5634	13.600	.2621	1.076	4.108	1.0027	4.128	.99729
85″51	12.818	3.3165	13.7821	13.818	.2591	1.075	4.150	1.0026	4.161	.99738
85″54	13.000	3.3309	13.9507	14.000	.2562	1.074	4.192	1.0026	4.203	.99744
85″58	13.217	3.3450	14.1821	14.217	.2534	1.073	4.235	1.0025	4.245	.99752
86″ 1	13.395	3.3590	14.3607	14.395	.2507	1.072	4.277	1.0024	4.287	.99758
86″ 4	13.578	3.3729	14.5438	14.578	.2480	1.071	4.318	1.0024	4.329	.99764
86″ 8	13.829	3.3866	14.7954	14.829	.2454	1.070	4.360	1.0023	4.370	.99772
86″11	14.023	3.4000	14.9898	15.023	.2429	1.069	4.402	1.0022	4.412	.99778
86″15	14.290		15.2570	15.290				1.0021		.99786
86″20	14.637		15.6048	15.637				1.0020		.99795
86″25	15.000		15.9687	16.000				1.0020		.99804
86″30	15.380		16.3499	16.380				1.0019		.99813
86″35	15.780		16.7496	16.780				1.0018		.99822
86″40	16.198		17.1693	17.198				1.0017		.99831
86″45	16.639		17.6106	17.639				1.0016		.99839
86″50	17.103		18.0750	18.103				1.0015		.99847
86″55	17.591		18.5645	18.591				1.0014		.99855
87″ 0	18.107		19.0811	19.107				1.0014		.99863
87″ 5	18.653		19.6273	19.653				1.0013		.99870
87″10	19.230		20.2055	20.230				1.0012		.99878
87″15	19.843		20.8188	20.843				1.0012		.99885
87″20	20.494		21.4704	21.494				1.0011		.99891
87″25	21.187		22.1640	22.187				1.0010		.99898
87″30	21.926		22.9038	22.926				1.0010		.99905
87″35	22.716		23.6945	23.716				1.0009		.99911
87″40	23.562		24.5418	24.562				1.0008		.99917
87″45	24.471		25.4517	25.471				1.0008		.99923
87″50	25.450		26.4316	26.450				1.0007		.99929
87″55	26.508		27.4900	27.508				1.0007		.99934
88″ 0	27.654		28.6362	28.654				1.0006		.99939
88″ 5	28.899		29.8823	29.899				1.0006		.99944
88″10	30.258		31.2416	31.258				1.0005		.99949
88″15	31.745		32.7303	32.745				1.0005		.99953
88″20	33.382		34.3678	34.382				1.0004		.99958
88″25	35.191		36.1776	36.191				1.0004		.99962
88″30	37.202		38.1885	38.202				1.0003		.99966
88″35	39.448		40.4358	40.448				1.0003		.99969
88″40	41.976		42.9641	42.976				1.0003		.99973
88″45	44.840		45.8293	45.840				1.0002		.99976
88″50	48.114		49.1039	49.114				1.0002		.99979
88″55	51.892		52.8821	52.892				1.0002		.99982

ϕ	x =sec. of ϕ—1	y	z =tan. of ϕ	t =sec. of ϕ	$\frac{y}{x}$	$\frac{z}{x}$	$\frac{z}{y}$	$\frac{t}{z}$ =cosec. of ϕ	$\frac{t}{y}$	$\frac{z}{t}$ =sine of ϕ
89„ 0	56.299		57.2900	57.299				1.0002		.99985
89„ 5	61.507		62.4991	62.507				1.0001		.99987
89„10	67.757		68.7501	68.757				1.0001		.99989
89„15	75.396		76.3900	76.396				1.0001		.99991
89„20	84.946		85.9398	85.946				1.0001		.99993
89„25	97.223		98.2179	98.223				1.0001		.99995
89„30	113.593		114.5886	114.593				1.0000		.99996
89„35	136.511		137.5074	137.511				1.0000		.99997
89„40	170.888		171.8854	171.888				1.0000		.99998
89„45	228.184		229.1817	229.184				1.0000		.99999
89„50	342.775		343.7737	343.775				1.0000		.99999
89„55	686.550		687.5489	687.550				1.0000		.99999
90„ 0			Infinite.	Infinite.				1.0000		1.0000